

Fast drift effects in the averaging of a filtration combustion system – a periodic homogenization approach

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Abstract

We target at the periodic homogenization of a semi-linear reaction-diffusion-convection system describing filtration combustion, where fast drifts affect the competition between heat and mass transfer processes as well as the interplay between the surface nonlinear chemical reactions and the transport processes. To handle the heterogeneity of the medium, we rely on the concept of two-scale convergence with drift to obtain for suitably scaled model parameters the upscaled system of combustion equations together with the effective transport and reaction parameters. The main difficulty here is to treat the case of the system combined with the nonlinearity of the surface production.

Key words. Filtration combustion, thermal dispersion, periodic homogenization, two-scale convergence with drift

MSC 80A25, 35B27, 76M50, 80A32

1 Introduction

Combustion of solid fuel in porous medium when air (or any other gaseous oxidizer) is injected has multiple applications in nowadays technology such as in self-propagating high-temperature synthesis [1], smoldering combustion in normal and microgravity environments [2, 3, 4], etc. The latter papers emphasize on the characteristic properties of smoldering combustion, specifically in microgravity environments where fire safety is of paramount importance. In this paper, we study the case of a dry porous medium, fully saturated by the presence of an oxidizing gaseous mixture in its gas phase, and possibly allowing for both fast drifts and fast combustion (fast gas-solid chemical reaction). The latter can be measured in the range of dominant Peclet and Damköhler numbers, which gives rise to dispersion in the proposed system.

The importance of dispersion phenomena in porous media combustion and other engineering applications is well-known in the literature; see, e.g., [5, 6, 7, 8]. In [9], thermal dispersion coefficients are calculated for an infinite porous medium by means of the volume averaging method; see also in this context [10, 11]. Furthermore, the homogenization method have been applied in the derivation of dispersion coefficients; see, e.g., [12, 13]. However, in

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the framework of filtration combustion, there is no detailed account of the dispersion effect on the transport parameters, which arises from the strong competition between mass and thermal transport in the presence of chemical reaction.

The reaction-diffusion-flow scenario of interest here can be briefly described as follows: A gaseous oxidizer is brought in by molecular diffusion superposed to a given drift¹ inside a porous medium made of periodic arrangements of pores; building a skeleton that covers \mathbb{R}^d . The oxidizer reacts with the solid fuel exothermically giving rise to high temperatures that travel further through the porous medium by means of heat conduction.²

We target at the upscaling of this combustion scenario, paying particular attention to capturing the effect of the microscopic drift, in the presence of dominant Damköhler numbers, on the governing macroscopic (upscaled) combustion equations, and specifically, on the structure of the effective thermal and mass dispersion tensors. The main mathematical difficulties arising in our context are fourfold:

- (i) the non-linearity of the gas-solid chemical reaction;
- (ii) the treatment of a coupled system of PDEs posed in high-contrasting microstructures;
- (iii) the fast drift;
- (iv) the coupled evolution system.

In (i), the nonlinearity is the temperature-dependent Arrhenius law, which poses a more mathematical difficulty, even in the simple form of a first order chemical kinetics. We will explore the full nonlinearity of the problem in our analysis by mainly relying on the structural properties of the nonlinearity. However, one possibility of avoiding the nonlinearity in the Arrhenius law is to consider its linear approximation as suggested in [14, 15]; nevertheless, the working hypothesis for such linear approximations is not so obvious. We borrow from our previous experience in handling situations like (ii) (see for instance [16, 17] for the treatment of a related scenario from chemical attack on concrete structures) and focus on the fast drift (iii), which, at first sight, is an impediment to the classical theory of homogenization and on the new aspect (iv). We adapt the working technique [18, 19] to our combustion setting, so that we can deal with the aspect (iii) in combination with (iv); see also [20, 21].

We foresee possible extensions to our approach, which is capable of handling locally periodic coverings of \mathbb{R}^d ; considerations on the bounded domain case being however out of reach for the moment. Due to finite-size effects, localizations of both heat and concentration seem unavoidable, at least for naive scalings, see e.g. [22] for evidence of localization in a scalar case.

The formal homogenization procedure of this combustion setting as well as extensive multiscale numerical simulations will be reported elsewhere. In a forthcoming publication, we will also look into the case when liquid islands are initially present in the porous medium, which typically occurs in coal gasification or in-situ combustion in oil recovery, see e.g. [23].

We describe the geometry of the porous medium in Section 2, while Section 2.1 contains the essence of our scaling arguments. The model equations are listed in Section 2.2. The main result of this paper is a set of upscaled equations and effective coefficients (for the

¹The drift can possibly be given as a posteriori approximation of the solution corresponding to an appropriate variant of compressible Navier-Stokes-like equations.

²We assume all radiation effects to be negligible.

heat capacity, transport and chemical reaction), summarized in Theorem 3.4 and reported in Section 3.1. The proof of this result is shown in Section 4.

2 Background on filtration combustion

Basic modeling considerations in combustion can be looked up, for instance, in [24, 25] and references cited therein. The pore scale description of the studied filtration model, which we refer to in this framework, has been previously introduced in [26, 27]. We recall here only the main ingredients of the model formulation.

We consider an infinite porous medium $\Omega \in \mathbb{R}^d$ made up of a solid phase (a simple material) that occupies a region Ω_s and a gas phase (saturated with an incompressible³ fluid containing a gaseous mixture that occupies a gaseous region Ω_g). The velocity of the fluid is assumed to be known and independent of the macroscopic space variable x and time variable t . The flow field satisfies a non penetrating condition on the gas-solid interphase $\partial\Omega$. We also assume that the gaseous mixture does not penetrate the solid part of the medium consisting of distributed inclusions of a reactive solid fuel, but chemically reacts with the solid at the gas-solid interphase. Here, in the simple model, we do not incorporate any kinetic model for the solid fuel, but we are however involved with a heterogeneous chemical reaction between the solid fuel and the gaseous oxidizer.

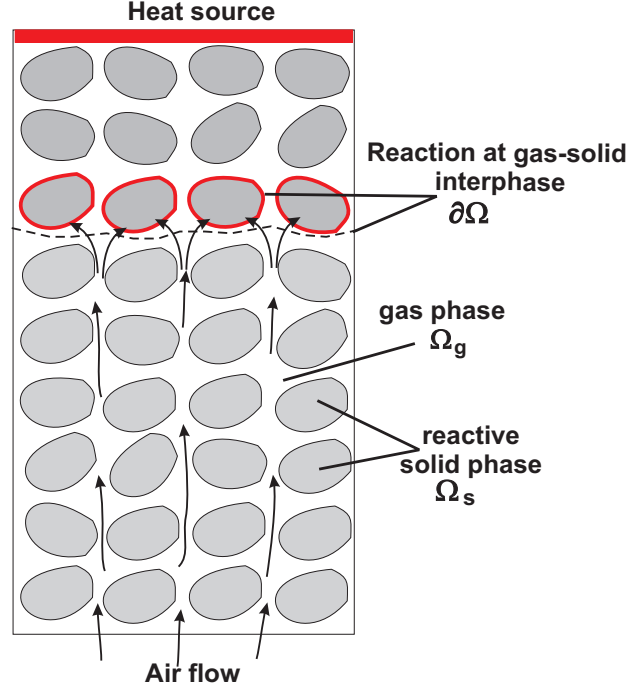


Figure 2.1: Schematic of the filtration combustion process at the pore scale.

The thermal conductivity and molecular diffusion tensors satisfy

$$\lambda(x) = \begin{cases} \lambda_g(x), & x \in \Omega_g \\ \lambda_s(x), & x \in \Omega_s \end{cases}, \quad \tilde{D}(x) = \begin{cases} D(x), & x \in \Omega_g \\ 0, & x \in \Omega_s, \end{cases} \quad (2.1)$$

³The incompressibility constraint is not essential. It can be removed, but calculations become more involved.

emphasizing, in this way, the contrast between the gas and solid phases. Following the same line, the velocity field and the volumetric heat capacities in the porous medium can be written as

$$\tilde{b}(x) = \begin{cases} b, & x \in \Omega_g \\ 0, & x \in \Omega_s \end{cases} \quad c(x) = \begin{cases} c_g(x), & x \in \Omega_g \\ c_s(x), & x \in \Omega_s. \end{cases} \quad (2.2)$$

Let the temperature in the whole domain Ω be denoted by $T(t, x)$ and the concentration of the gaseous oxidizer being transported by the fluid in the gas phase be denoted by $\mathcal{C}(t, x)$. Furthermore, we decompose $T(t, x)$ in the two subdomains as follows:

$$T(t, x) = \begin{cases} T_s(t, x), & x \in \Omega_s, \\ T_g(t, x), & x \in \Omega_g, \end{cases} \quad (2.3)$$

where $T_g(t, x)$ and $T_s(t, x)$ represents the temperatures in the gas phase and solid phase, respectively. Depending on the situation, it is sometimes convenient to work with the temperature $T(t, x)$, while at some other times the use of *two temperatures*⁴ is more convenient.

Denote by $T_f \in (0, \infty)$ the final time of the combustion process. The balance of heat and mass transport in the porous medium is given by two convection-diffusion-like equations posed in Ω_g and a heat conduction equation in Ω_s , viz.

$$\begin{cases} c_g \frac{\partial T_g}{\partial t} + c_g b \cdot \nabla T_g - \nabla \cdot (\lambda_g \nabla T_g) = 0, & \text{in } (0, T_f) \times \Omega_g, \\ c_s \frac{\partial T_s}{\partial t} - \nabla \cdot (\lambda_s \nabla T_s) = 0, & \text{in } (0, T_f) \times \Omega_s, \\ \frac{\partial \mathcal{C}}{\partial t} + b \cdot \nabla \mathcal{C} - \nabla \cdot (D \nabla \mathcal{C}) = 0, & \text{in } (0, T_f) \times \Omega_g. \end{cases} \quad (2.4)$$

The system (2.4) is coupled at the gas-solid interphase according to the following flux balances:

$$\lambda_g \nabla T \cdot n = \lambda_s \nabla T \cdot n + QW(T, \mathcal{C}), \quad \text{on } (0, T_f) \times \partial\Omega, \quad (2.5)$$

$$D \nabla \mathcal{C} \cdot n = -W(T, \mathcal{C}), \quad \text{on } (0, T_f) \times \partial\Omega, \quad (2.6)$$

where $Q > 0$ represents the heat release and $W(T, \mathcal{C})$ defined as

$$W(T, \mathcal{C}) := A \mathcal{C} f(T), \quad \text{with } f(T) = \exp\left(-\frac{T_a}{T}\right) \quad (2.7)$$

is a first-order Arrhenius kinetics with a pre-exponential factor A and an activation temperature T_a . The form of (2.7) agrees with a constant solid fuel assumption. We assume the continuity of temperature across the interface.

Now, we describe the structure of the porous medium of interest. We assume that Ω^ϵ is an ϵ -periodic unbounded open set of \mathbb{R}^d , $d \in \{2, 3\}$, which is subdivided into a periodic distribution of cells Y^ϵ , with each cell being equal and defined up to a scaled translation of a reference unit cell Y . The unit cell consists of two distinct parts—a gas-filled part and

⁴The decomposition of the temperature should not be confused with that one typically arising in the theory of heat conduction involving two temperatures (i.e. the *conductive* and the *thermodynamic* temperatures). In our scenario, since the solid phase is made up of a simple material, the conductive and thermodynamic temperatures coincide; compare [28].

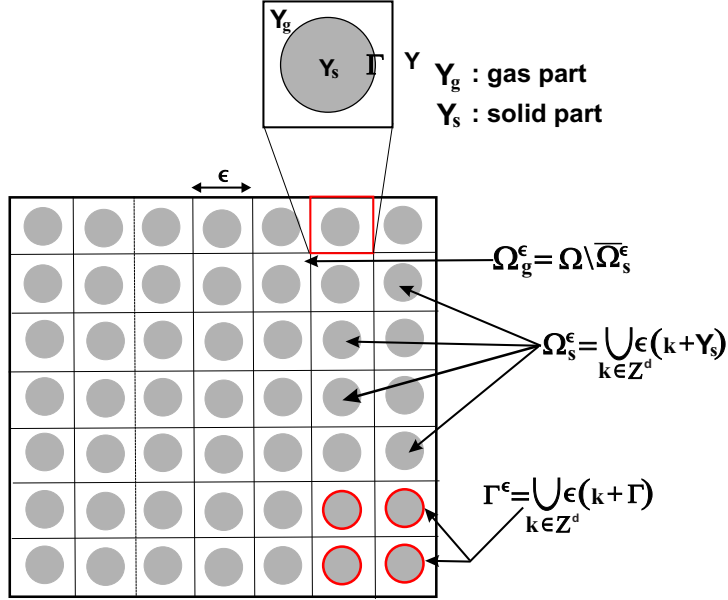


Figure 2.2: Two-dimensional periodic domain setting the stage for the homogenization and the reference cell. The solid inclusions are represented by the gray disks with the boundaries in red. The remaining part of the medium consists of the gas domain.

a solid part. We denote by Y_s , the solid part of Y having a smooth boundary denoted by Γ . The gas-filled part is a smooth connected open set defined by $Y_g = Y \setminus \overline{Y_s}$. We denote the collection of the periodically distributed solid parts by $Y_{s,k}^\epsilon = \epsilon(k + Y_s)$ and thus the periodically distributed boundaries is denoted by $\Gamma_k^\epsilon = \epsilon(k + \Gamma)$. Thus, the ensemble of the periodically translated solid parts, the matrix of connected gas-filled parts and the ensemble of the solid boundaries can be written as

$$\Omega_s^\epsilon = \bigcup_{k \in \mathbb{Z}^d} Y_{s,k}^\epsilon, \quad \Omega_g^\epsilon = \Omega^\epsilon \setminus \overline{\Omega_s^\epsilon}, \quad \Gamma^\epsilon = \bigcup_{k \in \mathbb{Z}^d} \Gamma_k^\epsilon, \quad (2.8)$$

For the ϵ -periodic representation of the porous medium, we assume that all involved physical parameters and functions are rapidly oscillating. That is, for a given function ψ , one may write $\psi^\epsilon(x) = \psi(x/\epsilon)$, for all macroscopic variable $x \in \Omega^\epsilon$ with $y = x/\epsilon$ representing the microscopic variable. We also assume that the function $\psi(y)$ and physical parameters can be extended by Y -periodicity to the whole \mathbb{R}^d (with a period length ϵ). Thus, the triplet (T, \mathcal{C}, b) can be written as $(T^\epsilon, C^\epsilon, b^\epsilon)$, pointing out this way the dependence of the solution vector on the ϵ -changes in the domain. We refer to [29] for a rigorous mathematical description of the geometry of the (periodic) porous media. Without loss of generality, we will sometimes use throughout the paper

$$\int_Y \psi(y) dy \quad \text{instead of} \quad \int_{Y_g} \psi_g(y) dy + \int_{Y_s} \psi_s(y) dy$$

for the functions and parameters defined as in (2.1), (2.2) and (2.3). Similar representations also hold for their counterparts defined in Ω^ϵ . In a slight abuse of notation, we will in particular refer to b^ϵ , instead of \tilde{b}^ϵ , as the extended velocity field when dealing with problems

defined in the domain Ω^ϵ . We also use the notation, $[\beta]_\Gamma = \beta_g - \beta_s$ on Γ , to represent the jump of a function, that takes values in the gas-filled region and solid region of the domain, across the boundary Γ and by $\psi(y)\Big|_\Gamma$, the restriction of the function ψ on Γ .

2.1 Scaling

Prior to performing the homogenization procedure, we normalize the system of governing equations as discussed in [26, 27]. The procedure leads to a couple of important dimensionless parameters. To obtain them, we introduce dimensionless variables as follows:

$$T^\epsilon = T^{\epsilon*}T_c, \quad C^\epsilon = C^{\epsilon*}C_c, \quad b^\epsilon = b^{\epsilon*}b_c, \quad x = x^*L, \quad t = t^*t_c, \quad (2.9)$$

where the subscript c denotes some constant characteristic quantity and the asterisk (*) denotes the corresponding dimensionless variable. We point out that similar characteristic quantities must be introduced for all physical quantities entering the equations, i.e. for any generic physical quantity ψ , the normalization scheme is given as $\psi = \psi^*\psi_c$, where ψ_c is some characteristic value of interest. The dimensionless parameters are derived from these characteristic quantities and are estimated in terms of orders of magnitude in ϵ ; see [30,]. Various choices of the estimates usually lead to different forms of the limit problem after the asymptotic procedure as $\epsilon \rightarrow 0$. The equations in their dimensionless forms can be written as:

$$\begin{cases} \mathcal{P}_T c_g^* \frac{\partial T_g^{\epsilon*}}{\partial t^*} + c_g^* \mathcal{P} e b^* \cdot \nabla T_g^{\epsilon*} - \nabla \cdot (\lambda_g^* \nabla T_g^{\epsilon*}) = 0, \\ \mathcal{P}_T m c_s^* \frac{\partial T_s^{\epsilon*}}{\partial t^*} - \mathcal{K} \nabla \cdot (\lambda_s^* \nabla T_s^{\epsilon*}) = 0, \\ \frac{\partial C^{\epsilon*}}{\partial t^*} + \mathcal{P} e b^* \cdot \nabla C^{\epsilon*} - \mathcal{L} e_g^{-1} \nabla \cdot (D^* \nabla C^{\epsilon*}) = 0, \\ \lambda_g^* \nabla T^{\epsilon*} \cdot n = \mathcal{K} \lambda_s^* \nabla T^{\epsilon*} \cdot n + \mathcal{D} a Q^* W^{\epsilon*}, \\ T_g^{\epsilon*} = T_s^{\epsilon*}, \\ D^* \nabla C^{\epsilon*} \cdot n = -\mathcal{L} e \mathcal{D} a W^{\epsilon*}, \end{cases} \quad (2.10)$$

where

$$W^{\epsilon*} := A^* C^{\epsilon*} \exp\left(-\frac{T_a^*}{T^{\epsilon*}}\right). \quad (2.11)$$

The problem introduces the following global characteristic time scales:

$$t_D := \frac{L^2}{D_c}, \quad t_A := \frac{L}{b_c}, \quad t_\lambda := \frac{c_{gc} L^2}{\lambda_{gc}}, \quad \text{and} \quad t_R := \frac{L}{A_c},$$

where t_D is the *characteristic global diffusion time scale*, t_A is the *characteristic global advection time scale*, t_λ is the *characteristic global time of conductive transfer*, while t_R is the *characteristic global chemical reaction time scale*.

We introduce additionally the following characteristic dimensionless numbers:

$$\begin{aligned} \mathcal{P} e &:= \frac{b_c L}{\alpha} = \frac{t_\lambda}{t_A} \quad (\text{Péclet number}), \\ \mathcal{L} e &:= \frac{\alpha}{D_c} = \frac{t_D}{t_\lambda} \quad (\text{Lewis number}), \\ \mathcal{D} a &:= \frac{A_c L}{\alpha} = \frac{t_\lambda}{t_R} \quad (\text{Damköhler number}), \end{aligned}$$

where $\alpha := \lambda_{gc}/c_{gc}$ is the *thermal diffusivity*. Other dimensionless quantities introduced in Eq. (2.10) include $\mathcal{P}_T = t_\lambda/t_c$, the ratio of *characteristic time of conductive transfer to the characteristic time scale of the observation*, $m = c_{sc}/c_{gc}$ is the *ratio of heat capacities*, $\mathcal{K} = \lambda_{sc}/\lambda_{gc}$ is the ratio of *heat conductivities*. We take the time of conductive heat transfer in the subdomain, Ω_g^ϵ , as the characteristic time of the observation at the macroscopic scale, i.e. $t_c = t_\lambda$. The characteristic temperature of the combustion product is given by $T_c = Q_c C_c / c_{gc}$ so that $T_a^* = T_a/T_c$ is the dimensionless activation temperature. To simplify the setting when passing to the homogenization limit, we assume that the constituents have heat capacities of the same order of magnitude, i.e., $m = \mathcal{O}(1)$. Since $t_c = t_\lambda$, it follows that $\mathcal{P}_T = \mathcal{O}(1)$. The Lewis number is considered in a regime in which the time of diffusion is comparable to the time of conductive heat transfer, i.e., $\mathcal{L}e = \mathcal{O}(1)$. We study a problem for which the constituent conductivities are of the same order of magnitude, i.e., $\mathcal{K} = \mathcal{O}(1)$. The regimes of interest are as follows:

$$\mathcal{P}e = \mathcal{O}(\epsilon^\gamma), \quad \mathcal{D}a = \mathcal{O}(\epsilon^\gamma), \quad \gamma = -1, 0, 1.$$

Note that, in principle, the asymptotic homogenization procedure can be performed (at least formally) for other combinations of scalings in terms of ϵ . However, when the Péclet and Damköhler numbers are not balanced, then the validity of our starting model (2.10) is restricted. For instance, high Péclet numbers in combination with moderate Damköhler numbers are prone to turbulence regimes (compare [31]), while high Damköhler numbers with moderate Péclet numbers most likely facilitate the occurrence of flames. In both such cases, the microscopic model has to be changed essentially. In one case, it becomes a system of stochastic partial differential equations, while in the other case the model stays at the deterministic level of description, but must include information about the *a priori* unknown position and velocity of the flames.

2.2 The microscopic problem

In this paper, we only study the fast regime, i.e. we focus on situations in which both the Péclet and Damköhler number are dominant. In other words, we assume (in line with [32]) that

$$\mathcal{P}e = \mathcal{O}(\epsilon^{-1}), \quad \mathcal{D}a = \mathcal{O}(\epsilon^{-1}). \quad (2.12)$$

Taking into account (2.12) and considering all the other parameters of the order of $\mathcal{O}(1)$ with respect to ϵ , we rewrite (2.10), after dropping the (*) and obtain the following microscopic

problem:

$$\left\{ \begin{array}{ll} c_g \frac{\partial T_g^\epsilon}{\partial t} + c_g \frac{b^\epsilon}{\epsilon} \cdot \nabla T_g^\epsilon - \nabla \cdot (\lambda_g \nabla T_g^\epsilon) = 0, & \text{in } (0, T_f) \times \Omega_g^\epsilon, \\ c_s \frac{\partial T_s^\epsilon}{\partial t} - \nabla \cdot (\lambda_s \nabla T_s^\epsilon) = 0, & \text{in } (0, T_f) \times \Omega_s^\epsilon, \\ \frac{\partial \mathcal{C}^\epsilon}{\partial t} + \frac{b^\epsilon}{\epsilon} \cdot \nabla \mathcal{C}^\epsilon - \nabla \cdot (D \nabla \mathcal{C}^\epsilon) = 0, & \text{in } (0, T_f) \times \Omega_g^\epsilon, \\ \lambda_g \nabla T^\epsilon \cdot n = \lambda_s \nabla T^\epsilon \cdot n + Q \frac{W^\epsilon}{\epsilon}, \\ T_g^\epsilon = T_s^\epsilon, \\ D \nabla \mathcal{C}^\epsilon \cdot n = -\frac{W^\epsilon}{\epsilon}, \\ \mathcal{C}^\epsilon(x, 0) = \mathcal{C}_0(x) \text{ for } x \in \Omega_g^\epsilon, \\ T^\epsilon(0, x) = T_0(x) \text{ for } x \in \Omega^\epsilon. \end{array} \right\} \quad \text{in } (0, T_f) \times \Gamma^\epsilon. \quad (2.13)$$

We refer to this system of equations as Problem \mathcal{P}^ϵ .

Remark 2.1. Interestingly, performing in the dimensional model (2.10) the change of variables $x = \epsilon y$ and $t = \epsilon^2 \tau$ leads to the same scaling in ϵ as it is shown in (2.13).

2.3 Assumptions. Concept of solution. Technical preliminaries

We denote by $S := (0, T_f)$ the time interval of observation of the combustion process with T_f , the final time. We take into account the following assumptions on the model parameters:

- (H1) $c_g, c_s, \lambda_g, \lambda_s, D, Q, A \in (0, \infty)$;
- (H2) The velocity field $b^\epsilon(x) = b\left(\frac{x}{\epsilon}\right)$ is periodic, bounded and divergence free, i.e. b is Y -periodic, $|b(y)| \in L^\infty(Y)$, $\text{div}_y b(y) = 0$ in Y_g and $b \cdot n|_\Gamma = 0$;
- (H3) The initial data are non-negative, i.e., $\mathcal{C}_0 \geq 0, T_0 > 0$ and $\mathcal{C}_0 \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $T_0 \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.
- (H4) The nonlinear function $f \in C^\infty(\mathbb{R})$ is monotonic and positive. A direct consequence of (H4) is the following exponentiation property:

$$f \geq f^2 \geq \dots \geq f^n, \quad n \in \mathbb{N}. \quad (2.14)$$

Remark 2.2. The assumptions (H1) and (H3) are physically motivated in the sense that heat capacities, heat conductivities and diffusion coefficients for isotropic materials are strictly positive real constants. On the other hand, (H2) is a mathematical restriction which strongly delimits the applicability of the method. For instance, note that if the velocity field b^ϵ would come from the background fluid satisfying the compressible Navier-Stokes equations, then it would automatically depend on both variables x and x/ϵ and hence this approach would not be applicable anymore in a straightforward fashion; see also Remark 2 in [20]. Assumption (H4) is a generally accepted form of the non-linearity in the combustion engineering community.

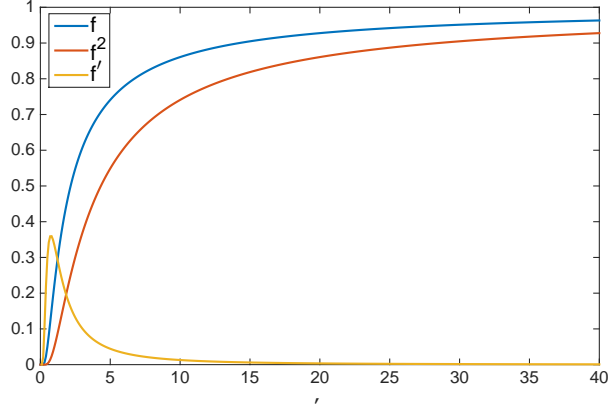


Figure 2.3: The profile of the function f and its first derivative and exponentiation property given by (2.14).

We introduce the function space

$$V^\epsilon := \{(\phi, \psi) \in H^1(\Omega) \times H^1(\Omega_g^\epsilon)\}$$

to work with the equivalent concept of solution as stated in the next Definition.

Definition 2.1. Assuming (H4), the couple $(T^\epsilon, \mathcal{C}^\epsilon)$ is called a weak solution to \mathcal{P}^ϵ if for all $t \in S$ and $(T^\epsilon(\cdot, 0), \mathcal{C}^\epsilon(\cdot, 0)) = (T_0, \mathcal{C}_0)$, the following identities hold

$$\int_{\Omega^\epsilon} c \frac{\partial T^\epsilon}{\partial t} \phi \, dx + \frac{1}{\epsilon} \int_{\Omega^\epsilon} c_g b^\epsilon \cdot \nabla T^\epsilon \phi \, dx + \int_{\Omega^\epsilon} \lambda \nabla T^\epsilon \cdot \nabla \phi \, dx - \frac{QA}{\epsilon} \int_{\Gamma^\epsilon} W^\epsilon \phi \, d\sigma = 0, \quad (2.15)$$

$$\int_{\Omega_g^\epsilon} \frac{\partial \mathcal{C}^\epsilon}{\partial t} \psi \, dx + \frac{1}{\epsilon} \int_{\Omega_g^\epsilon} b^\epsilon \cdot \nabla \mathcal{C}^\epsilon \nabla \psi \, dx + \int_{\Omega_g^\epsilon} D \nabla \mathcal{C}^\epsilon \cdot \nabla \psi \, dx + \frac{A}{\epsilon} \int_{\Gamma^\epsilon} W^\epsilon \psi \, d\sigma = 0 \quad (2.16)$$

for all test functions $(\phi, \psi) \in V^\epsilon$.

Lemma 2.1. Let $(T^\epsilon, \mathcal{C}^\epsilon)$ be a weak solution to (2.15) and (2.16) and assume (H1)–(H4). There exists a constant $C > 0$, which is independent of the choice of ϵ , such that the following a priori bounds is satisfied

$$\|T^\epsilon\|_{L^\infty(S; L^2(\Omega^\epsilon))} + \|\mathcal{C}^\epsilon\|_{L^\infty(S; L^2(\Omega_g^\epsilon))} + \|\nabla T^\epsilon\|_{L^2(S \times \Omega^\epsilon)} + \|\nabla \mathcal{C}^\epsilon\|_{L^2(S \times \Omega_g^\epsilon)} \quad (2.17)$$

$$+ \sqrt{\epsilon} \|\mathcal{W}^\epsilon\|_{L^\infty(S; L^2(\Gamma^\epsilon))} \leq C \left(\|T_0\|_{L^2(\mathbb{R}^d)} + \|\mathcal{C}_0\|_{L^2(\mathbb{R}^d)} \right), \quad (2.18)$$

where $\mathcal{W}^\epsilon = \frac{1}{\epsilon} \mathcal{C}^\epsilon f(T^\epsilon)$.

Proof. The desired a priori bounds can be obtained by testing (2.15) and (2.16) by $(\phi, \psi) =$

$(T^\epsilon, QC^\epsilon)$ and combining the resulting expression, i.e.

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|T^\epsilon(t)\|_{L^2(\Omega^\epsilon)}^2 + \frac{1}{\epsilon} (c_g b^\epsilon \nabla T^\epsilon, T^\epsilon) + \|\nabla T^\epsilon(t)\|_{L^2(\Omega^\epsilon)}^2 \\
& + \frac{Q}{2} \frac{d}{dt} \|C^\epsilon(t)\|_{L^2(\Omega_g^\epsilon)}^2 + \frac{Q}{\epsilon} (b^\epsilon \nabla C^\epsilon, C^\epsilon) + \frac{Q}{2} \|\nabla C^\epsilon(t)\|_{L^2(\Omega_g^\epsilon)}^2 \\
& + \frac{QA}{\epsilon} \int_{\Gamma^\epsilon} C^\epsilon f(T^\epsilon) (C^\epsilon - T^\epsilon) d\sigma = 0.
\end{aligned} \tag{2.19}$$

The convection terms vanishes, i.e.

$$\int_{\Omega^\epsilon} c_g b^\epsilon \cdot \nabla T^\epsilon T^\epsilon dx = \int_{\Omega_g^\epsilon} b^\epsilon \cdot \nabla C^\epsilon C^\epsilon dx = 0$$

since by (H2), we have that

$$\begin{aligned}
\int_{\Omega_g^\epsilon} b^\epsilon \cdot \nabla C^\epsilon C^\epsilon dx &= \frac{1}{2} \int_{\Omega_g^\epsilon} \operatorname{div} (b^\epsilon |C^\epsilon|^2) dx = -\frac{1}{2} \int_{\partial\Omega^\epsilon} |C^\epsilon|^2 b^\epsilon \cdot n d\sigma \\
&= -\frac{1}{2} \int_{\partial\Omega^\epsilon - \Gamma^\epsilon} |C^\epsilon|^2 b^\epsilon \cdot n d\sigma - \frac{1}{2} \int_{\Gamma^\epsilon} |C^\epsilon|^2 b^\epsilon \cdot n d\sigma = 0
\end{aligned}$$

Integrating (2.19) with respect to time, leads to the following inequality

$$\begin{aligned}
& \|T^\epsilon(t)\|_{L^\infty(S; L^2(\Omega^\epsilon))}^2 + \|\nabla T^\epsilon(t)\|_{L^2(S \times \Omega^\epsilon)}^2 \\
& + \|C^\epsilon(t)\|_{L^\infty(S; L^2(\Omega_g^\epsilon))}^2 + \|\nabla C^\epsilon(t)\|_{L^2(S \times \Omega_g^\epsilon)}^2 \\
& + \frac{QA}{\epsilon} \int_S \int_{\Gamma^\epsilon} (C^\epsilon)^2 f(T^\epsilon) d\sigma dt - \frac{QA}{\epsilon} \int_S \int_{\Gamma^\epsilon} T^\epsilon C^\epsilon f(T^\epsilon) d\sigma dt \\
& \leq C \left(\|T_0\|_{L^2(\mathbb{R}^d)}^2 + \|C_0\|_{L^2(\mathbb{R}^d)}^2 \right).
\end{aligned} \tag{2.20}$$

By (H4), we apply the property $f \geq f^2$ on the first nonlinear term and the positivity of f in the last term on the left hand side of (2.20). Thus, we arrive at the desired a priori bounds. \square

Restriction to a compact subset of \mathbb{R}^d Since the problem is posed in an unbounded domain, Rellich Theorem does not apply, and hence there is no hope of establishing the compactness of the sequence of functions (T^ϵ, C^ϵ) . Thus, we first restrict the problem to a compact subset of \mathbb{R}^d . However, due to the presence of the dominant convection in the problem, nontrivial convergence of the family of solutions $T^\epsilon(t, x)$ and $C^\epsilon(t, x)$ is not viable in fixed coordinates x . Thus, we will follow the argument in [19, 18] and prove compactness results in the moving coordinates $(x + \epsilon^{-1} b_T, x + \epsilon^{-1} b_C)$. To work in the latter frame of reference, we will introduce functions in the new reference frame, following a similar notation as described in [18]: For any given function $u^\epsilon(t, x)$ in the fixed coordinate frame, we define its counterpart in the moving coordinates as

$$\hat{u}^\epsilon(t, x) = u^\epsilon\left(t, x + \frac{b^*}{\epsilon} t\right), \tag{2.21}$$

where b^* represents either of the effective drift velocities, b_T or b_C , defined by (3.6) or (3.7) in Section 3.1. For drifts moving in the opposite direction to the direction of (2.21), (cf. Section 3 for the definition of the two-scale convergence with drift.), we will use

$$\check{u}^\epsilon(t, x) = u^\epsilon\left(t, x - \frac{b^*}{\epsilon}t\right). \quad (2.22)$$

Additionally, a consequence of (2.21) (respectively (2.22)) is to characterize the motion of the porous medium in the moving coordinates. For that, we assume the motion of the domain depends on the drift of the governing physical process, i.e., for the thermal dispersion in the porous medium, we write

$$\hat{\Omega}^\epsilon = \left\{ x + \frac{b_T}{\epsilon}t \mid x \in \Omega^\epsilon \right\}, \quad (2.23)$$

and for mass dispersion in the gaseous phase of the porous medium, we will use the definition

$$\hat{\Omega}_g^\epsilon = \left\{ x + \frac{b_C}{\epsilon}t \mid x \in \Omega_g^\epsilon \right\}. \quad (2.24)$$

Remark 2.3. *For the case of filtration combustion of gases, i.e. without a thermal spread in the solid phase, the disparity in the moving coordinates, given by (2.21)–(2.24), no longer applies since in such a case $b_T = b_C$.*

Lemma 2.2. *Let (T^ϵ, C^ϵ) be the solution of \mathcal{P}^ϵ . Then, for any $\delta > 0$, there exists $R(\delta) > 0$ such that for all $t \in \bar{S}$*

$$\|\hat{T}^\epsilon(t, x)\|_{L^2(\Omega^\epsilon(t) \cap (\mathbb{R}^d \setminus Q_{R(\delta)}))} \leq \delta, \quad \|\hat{C}^\epsilon(t, x)\|_{L^2(\Omega_g^\epsilon(t) \cap (\mathbb{R}^d \setminus Q_{R(\delta)}))} \leq \delta, \quad (2.25)$$

where

$$Q_{R(\delta)} =] - R(\delta), R(\delta) [^d \in \mathbb{R}^d. \quad (2.26)$$

Proof. The proof goes in a similar manner as in [18]. We introduce a smooth cut-off function $\psi \in C^\infty(\mathbb{R})$ satisfying $0 \leq \psi(r) \leq 1$, $\psi = 0$ for $r \leq 1$, $\psi = 1$ for $r \geq 2$. Then, for $x \in \mathbb{R}^d$, we denote $\psi_R(x) = \psi(|x|/R)$ and choose as test functions $(T^\epsilon \check{\psi}_{R,T}, Q C^\epsilon \check{\psi}_{R,C})$, in the variational formulation (2.15)–(2.16), where

$$\check{\psi}_{R,T}(t, x) = \psi_R\left(x - \frac{b_T}{\epsilon}t\right), \quad \check{\psi}_{R,C}(t, x) = \psi_R\left(x - \frac{b_C}{\epsilon}t\right).$$

Next, we integrate by parts in time, the terms with the time derivatives

$$\begin{aligned} \int_0^t \int_{\Omega^\epsilon} c \frac{\partial T^\epsilon}{\partial t} T^\epsilon \check{\psi}_{R,T} \, dx ds &= \frac{1}{2\epsilon} \int_0^t \int_{\Omega^\epsilon} c |T^\epsilon|^2 b_T \cdot \nabla \check{\psi}_{R,T} \, dx ds \\ &\quad - \int_{\Omega^\epsilon} c |T^\epsilon(t, x)|^2 \check{\psi}_{R,T}(t, x) \, dx - \int_{\Omega^\epsilon} c |T_0|^2 \check{\psi}_R(x) \, dx \end{aligned}$$

and

$$\begin{aligned} Q \int_0^t \int_{\Omega_g^\epsilon} \frac{\partial \mathcal{C}^\epsilon}{\partial t} \check{\psi}_{R,C} \, dx ds &= \frac{Q}{2\epsilon} \int_0^t \int_{\Omega_g^\epsilon} |\mathcal{C}^\epsilon|^2 b_C \cdot \nabla \check{\psi}_{R,C} \, dx ds \\ Q \int_{\Omega_g^\epsilon} |\mathcal{C}^\epsilon(t, x)|^2 \check{\psi}_{R,C}(t, x) \, dx &- Q \int_{\Omega_g^\epsilon} |\mathcal{C}_0|^2 \check{\psi}_R(x) \, dx. \end{aligned}$$

In turn, we apply integration by parts on the terms with the space derivatives. Starting with the convection terms, we obtain

$$\frac{1}{\epsilon} \int_0^t \int_{\Omega^\epsilon} c_g b^\epsilon \cdot \nabla T^\epsilon T^\epsilon \check{\psi}_{R,T} \, dx ds = -\frac{1}{2\epsilon} \int_0^t \int_{\Omega^\epsilon} c_g |T^\epsilon|^2 b^\epsilon \cdot \nabla \check{\psi}_{R,T} \, dx ds$$

and

$$\frac{Q}{\epsilon} \int_0^t \int_{\Omega_g^\epsilon} b^\epsilon \cdot \nabla \mathcal{C}^\epsilon \mathcal{C}^\epsilon \check{\psi}_{R,C} \, dx ds = -\frac{Q}{2\epsilon} \int_0^t \int_{\Omega_g^\epsilon} |\mathcal{C}^\epsilon|^2 b^\epsilon \cdot \nabla \check{\psi}_{R,C} \, dx ds$$

and then the diffusive terms

$$\begin{aligned} - \int_0^t \int_{\Omega^\epsilon} \operatorname{div}(\lambda \nabla T^\epsilon) T^\epsilon \check{\psi}_{R,T} \, dx ds &= \int_0^t \int_{\Omega^\epsilon} \lambda \nabla T^\epsilon \cdot \nabla T^\epsilon \check{\psi}_{R,T} \, dx ds \\ + \frac{1}{2} \int_0^t \int_{\Omega^\epsilon} \lambda \nabla |T^\epsilon|^2 \nabla \check{\psi}_{R,T} \, dx ds &- \frac{QA}{\epsilon} \int_0^t \int_{\Gamma^\epsilon} \mathcal{C}^\epsilon T^\epsilon f(T^\epsilon) \check{\psi}_{R,T} \, d\sigma ds \end{aligned}$$

and

$$\begin{aligned} - Q \int_0^t \int_{\Omega_g^\epsilon} \operatorname{div}(D \nabla \mathcal{C}^\epsilon) \mathcal{C}^\epsilon \check{\psi}_{R,C} \, dx ds &= Q \int_0^t \int_{\Omega_g^\epsilon} D \nabla \mathcal{C}^\epsilon \cdot \nabla \mathcal{C}^\epsilon \check{\psi}_{R,C} \, dx ds \\ + \frac{Q}{2} \int_0^t \int_{\Omega_g^\epsilon} D \nabla |\mathcal{C}^\epsilon|^2 \nabla \check{\psi}_{R,C} \, dx ds &+ \frac{QA}{\epsilon} \int_0^t \int_{\Gamma^\epsilon} |\mathcal{C}^\epsilon|^2 f(T^\epsilon) \check{\psi}_{R,C} \, d\sigma ds. \end{aligned}$$

By applying (H4) and combining all the terms together, we have

$$\int_{\Omega^\epsilon} |T^\epsilon(t, x)|^2 \check{\psi}_{R,T}(t, x) dx + \int_0^t \int_{\Omega^\epsilon} \lambda \nabla T^\epsilon \cdot \nabla T^\epsilon \check{\psi}_{R,T} dx ds \quad (2.27)$$

$$+ Q \int_{\Omega_g^\epsilon} |\mathcal{C}^\epsilon(t, x)|^2 \check{\psi}_{R,\mathcal{C}}(t, x) dx + Q \int_0^t \int_{\Omega_g^\epsilon} D \nabla \mathcal{C}^\epsilon \cdot \nabla \mathcal{C}^\epsilon \check{\psi}_{R,\mathcal{C}} dx ds + \frac{QA}{\epsilon} \int_0^t \int_{\Gamma^\epsilon} |\mathcal{C}^\epsilon f(T^\epsilon)|^2 \check{\psi}_{R,\mathcal{C}} d\sigma ds \quad (2.28)$$

$$\leq \frac{1}{2\epsilon} \int_0^t \int_{\Omega^\epsilon} c |T^\epsilon|^2 (b^\epsilon - b_T) \cdot \nabla \check{\psi}_{R,T} dx ds + \frac{Q}{2\epsilon} \int_0^t \int_{\Omega_g^\epsilon} |\mathcal{C}^\epsilon|^2 (b^\epsilon - b_C) \cdot \nabla \check{\psi}_{R,\mathcal{C}} dx ds \quad (2.29)$$

$$- \frac{1}{2} \int_0^t \int_{\Omega^\epsilon} \lambda \nabla |T^\epsilon|^2 \nabla \check{\psi}_{R,T} dx ds - \frac{Q}{2} \int_0^t \int_{\Omega_g^\epsilon} D \nabla |\mathcal{C}^\epsilon|^2 \nabla \check{\psi}_{R,\mathcal{C}} dx ds \quad (2.30)$$

$$+ \int_{\Omega^\epsilon} c |T_0|^2 \check{\psi}_R(x) dx + \int_{\Omega_g^\epsilon} |\mathcal{C}_0|^2 \check{\psi}_R(x) dx. \quad (2.31)$$

To deal with the singular nature of the convective terms on the right hand side of (2.29), we introduce two auxiliary problems:

$$\begin{cases} -\Delta \Pi_i(y) = c(y)(b_{T,i} - b_i(y)) & \text{in } Y, \\ -\nabla \Pi_i \cdot n = 0 & \text{on } \Gamma, \\ \Pi_i(y) \text{ is } Y\text{-periodic} \end{cases} \quad \begin{cases} -\Delta \Sigma_i(y) = b_{C,i} - b_i(y) & \text{in } Y_g, \\ -\nabla \Sigma_i \cdot n = 0 & \text{on } \Gamma, \\ \Sigma_i(y) \text{ is } Y\text{-periodic}, \end{cases} \quad (2.32)$$

where the solutions Π_i and Σ_i are unique up to additive constants, by virtue of definitions (3.6) and (3.7) of the effective drift constants. Since $\epsilon \nabla \Pi_i^\epsilon(x) = (\nabla_y \Pi_i)(x/\epsilon)$ (similarly for the variable Σ), periodic extensions of (2.32) in \mathbb{R}^d allow us to write:

$$\begin{cases} -\epsilon^2 \Delta \Pi_i^\epsilon(x) = c^\epsilon(x)(b_{T,i} - b_i^\epsilon(x)) & \text{in } \Omega^\epsilon, \\ -\nabla \Pi_i^\epsilon \cdot n = 0 & \text{on } \Gamma^\epsilon, \\ \Pi_i^\epsilon(x) \text{ is } \epsilon\text{-periodic} \end{cases} \quad \begin{cases} -\epsilon^2 \Delta \Sigma_i^\epsilon(x) = b_{C,i} - b_i^\epsilon(x) & \text{in } \Omega_g^\epsilon, \\ -\nabla \Sigma_i^\epsilon \cdot n = 0 & \text{on } \Gamma^\epsilon, \\ \Sigma_i^\epsilon(x) \text{ is } \epsilon\text{-periodic}. \end{cases} \quad (2.33)$$

Remark 2.4. For brevity of presentation, we have assumed suitable extensions for the velocity field and parameters defined in Y , respectively in Ω^ϵ . In essence, (2.1) and (2.2) apply in situations just as in (2.32) and (2.33) by introducing distinct problems in the solid and gas-filled regions of Y as follows:

$$\begin{cases} -\Delta \Pi_{g,i}(y) = c_g(b_{T,i} - b_i(y)) & \text{in } Y_g, \\ -\Delta \Pi_{s,i}(y) = c_s b_{T,i} & \text{in } Y_s, \\ (\nabla \Pi_{g,i} - \nabla \Pi_{s,i}) \cdot n = 0 & \text{on } \Gamma, \\ \Pi_i(y) \text{ is } Y\text{-periodic}. \end{cases} \quad (2.34)$$

Substituting (2.33) in (2.29) and integrating by part

$$\sum_{i=1}^d \int_0^t \left(\int_{\Omega^\epsilon} \epsilon \nabla \Pi_i^\epsilon \cdot \nabla \left(|T^\epsilon|^2 \partial_{x_i} \check{\psi}_{R,T} \right) dx + \int_{\Omega^\epsilon} \epsilon \nabla \Sigma_i^\epsilon \cdot \nabla \left(|\mathcal{C}^\epsilon|^2 \partial_{x_i} \check{\psi}_{R,\mathcal{C}} \right) dx \right) ds.$$

By definition of ψ_R , we have $\|\nabla \check{\psi}_R\|_{L^\infty(\Omega^\epsilon)} \leq C/R$. Thus, the respective terms in (2.30) can be bounded as follows

$$\frac{C}{R} \left(\|\nabla T^\epsilon\|_{L^2(S \times \Omega^\epsilon)} + \|\nabla \mathcal{C}^\epsilon\|_{L^2(S \times \Omega^\epsilon)} \right) \leq \frac{C}{R}. \quad (2.35)$$

The last inequality in (2.35) is a consequence of Lemma 2.1, which also imply that (2.29) is bounded by C/R . For sufficiently large R , (2.31) tends to zero. Eventually, a simple change of coordinate and for sufficiently large R we arrive at the desired result. \square

Equicontinuity in time for the sequences of functions $(T^\epsilon, \mathcal{C}^\epsilon)$ In this step, we establish an equicontinuity in time for the sequences of functions $(T^\epsilon, \mathcal{C}^\epsilon)$ in the moving coordinates. For this, we introduce the orthonormal basis $\{e_j\}_{j \in \mathbb{Z}^d} \in L^2([0, 1]^d)$, such that $e_j \in C_0^\infty([0, 1]^d)$. Then the functions $\{e_{jk}\}_{j,k \in \mathbb{Z}^d}$, where $e_{jk}(x) = e_j(x-k)$, form an orthonormal basis in $L^2(\mathbb{R}^d)$.

Lemma 2.3. *Let $\delta t > 0$ represent a small parameter for translation in time. Then there exists a positive constant C_{jk} independent of ϵ and δt such that*

$$\left| (\hat{T}^\epsilon(t + \delta t, \cdot), e_{jk})_{L^2(\hat{\Omega}^\epsilon(t + \delta t))} - (\hat{T}^\epsilon(t, \cdot), e_{jk})_{L^2(\hat{\Omega}^\epsilon(t))} \right| \leq C_{jk} \sqrt{\delta t} \quad (2.36)$$

$$\left| (\hat{\mathcal{C}}^\epsilon(t + \delta t, \cdot), e_{jk})_{L^2(\hat{\Omega}_g^\epsilon(t + \delta t))} - (\hat{\mathcal{C}}^\epsilon(t, \cdot), e_{jk})_{L^2(\hat{\Omega}_g^\epsilon(t))} \right| \leq C_{jk} \sqrt{\delta t}. \quad (2.37)$$

Proof. Let us denote

$$\check{e}_{jk}(t, x) = e_{jk}\left(x - \frac{b_i}{\epsilon} t\right), \quad i = T, \mathcal{C}$$

and use $(\check{e}_{jk}, Q\check{e}'_{jk})$ as a test function in the variational formulation of \mathcal{P}^ϵ . First, we start with the sequences of functions defined in the moving coordinates and compute the difference

$$\begin{aligned} & \left(\hat{T}^\epsilon(t + \delta t, x), e_{jk}(x) \right)_{L^2(\hat{\Omega}^\epsilon(t + \delta t))} - \left(\hat{T}^\epsilon(t, x), e_{jk}(x) \right)_{L^2(\hat{\Omega}^\epsilon(t))} \\ & + Q \left(\hat{\mathcal{C}}^\epsilon(t + \delta t, x), e_{jk}(x) \right)_{L^2(\hat{\Omega}_g^\epsilon(t + \delta t))} - Q \left(\hat{\mathcal{C}}^\epsilon(t, x), e_{jk}(x) \right)_{L^2(\hat{\Omega}_g^\epsilon(t))} \\ & = \int_t^{t+\delta t} \frac{d}{ds} \left\{ \int_{\hat{\Omega}^\epsilon(s)} \hat{T}^\epsilon(s, x) e_{jk}(x) dx + Q \int_{\hat{\Omega}_g^\epsilon(s)} \hat{\mathcal{C}}^\epsilon(s, x) e_{jk}(x) dx \right\}. \end{aligned} \quad (2.38)$$

In the next step, we change to the fixed coordinates by setting $x \mapsto x - b_i t/\epsilon$ on the right

hand side of (2.38). Then, we recall the variational formulation in fixed coordinates

$$= \int_t^{t+\delta t} \int_{\Omega^\epsilon} \left\{ \frac{\partial T^\epsilon}{\partial s}(s, x) \check{e}_{jk}(x) - \frac{b_T}{\epsilon} \cdot \nabla \check{e}_{jk}(x) T^\epsilon(s, x) \right\} dx ds \quad (2.39)$$

$$+ Q \int_t^{t+\delta t} \int_{\Omega_g^\epsilon} \left\{ \frac{\partial \mathcal{C}^\epsilon}{\partial s}(s, x) \check{e}'_{jk}(x) - \frac{b_C}{\epsilon} \cdot \nabla \check{e}'_{jk}(x) \mathcal{C}^\epsilon(s, x) \right\} dx ds$$

$$= \int_t^{t+\delta t} \int_{\Omega^\epsilon} \left(\frac{b^\epsilon - b_T}{\epsilon} \right) \cdot \nabla \check{e}_{jk}(x) T^\epsilon(s, x) dx ds - \int_t^{t+\delta t} \int_{\Omega^\epsilon} \lambda \nabla T^\epsilon \cdot \nabla \check{e}_{jk}(x) dx ds$$

$$+ Q \int_t^{t+\delta t} \int_{\Omega_g^\epsilon} \left(\frac{b^\epsilon - b_C}{\epsilon} \right) \cdot \nabla \check{e}'_{jk}(x) \mathcal{C}^\epsilon(s, x) dx ds - Q \int_t^{t+\delta t} \int_{\Omega_g^\epsilon} D \nabla \mathcal{C}^\epsilon \cdot \nabla \check{e}'_{jk}(x) dx ds \quad (2.40)$$

$$+ \frac{QA}{\epsilon} \int_t^{t+\delta t} \int_{\Gamma^\epsilon} \mathcal{C}^\epsilon f(T^\epsilon) (\check{e}_{jk} - \check{e}'_{jk}) d\sigma ds$$

$$= \int_t^{t+\delta t} \int_{\Omega^\epsilon} \left\{ \epsilon \Delta \Pi_i^\epsilon(x) \partial_{x_i} \check{e}_{jk} T^\epsilon(s, x) - \lambda \nabla T^\epsilon(s, x) \cdot \nabla \check{e}_{jk}(x) \right\} dx ds \quad (2.41)$$

$$+ \int_t^{t+\delta t} \int_{\Omega_g^\epsilon} \left\{ \epsilon \Delta \Sigma_i^\epsilon(x) \partial_{x_i} \check{e}'_{jk} \mathcal{C}^\epsilon(s, x) - D \nabla \mathcal{C}^\epsilon(s, x) \cdot \nabla \check{e}'_{jk}(x) \right\} dx ds \quad (2.42)$$

$$+ \frac{QA}{\epsilon} \int_t^{t+\delta t} \int_{\Gamma^\epsilon} \mathcal{C}^\epsilon f(T^\epsilon) (\check{e}_{jk} - \check{e}'_{jk}) d\sigma ds. \quad (2.43)$$

Note that the singularity of the convective terms has been dealt with by using the auxiliary equations (2.33). In order to handle the nonlinear integral (2.43) with the coefficient ϵ^{-1} , we consider the following auxiliary problem

$$\begin{cases} \operatorname{div}_y^s \Upsilon(y) = \phi(t, y) = \check{e}_{jk}(t, y) - \check{e}'_{jk}(t, y), & \text{on } \Gamma \\ \Upsilon(y) \text{ is } Y\text{-periodic,} \end{cases} \quad (2.44)$$

where the vector field $\Upsilon(y)$ possesses a unique solution in $H_{\#}^1(\Gamma)/\mathbb{R}$ provided that

$$\int_{\Gamma} (\check{e}_{jk} - \check{e}'_{jk})(t, y) d\sigma = 0, \text{ for a.e. } t \in S. \quad (2.45)$$

Here, $\operatorname{div}_y^s = \operatorname{div}_y^s M(y)$ is the tangential divergence with $M(y) = \operatorname{Id} - n(y) \otimes n(y)$, the projection matrix on the tangent hyperplane to the surface Γ . Note that we have assumed the restriction of the basis functions \check{e}_{jk} on Y by the change of variable $x = \epsilon y$, which immediately

transforms (2.44) to

$$\begin{cases} \epsilon \operatorname{div}_x^s \Upsilon^\epsilon(x) = \phi(t, x) = \check{e}_{jk}(t, x) - \check{e}'_{jk}(t, x), & \text{on } \Gamma^\epsilon \\ \Upsilon^\epsilon(x) \text{ is } \epsilon\text{-periodic.} \end{cases} \quad (2.46)$$

Integrating (2.41) and (2.42) by parts, applying (2.46) on (2.43) and introducing the nonlinear term $\mathcal{W}^\epsilon = \epsilon^{-1} \mathcal{C}^\epsilon f(T^\epsilon)$, we have

$$= - \int_t^{t+\delta t} \int_{\Omega^\epsilon} \left\{ \nabla_y \Pi_i(y) \cdot \nabla (\partial_{x_i} \check{e}_{jk} T^\epsilon(s, x)) + \lambda \nabla T^\epsilon(s, x) \cdot \nabla \check{e}_{jk}(x) \right\} dx ds \quad (2.47)$$

$$- \int_t^{t+\delta t} \int_{\Omega_g^\epsilon} \left\{ \nabla_y \Sigma_i^\epsilon(y) \cdot \nabla (\partial_{x_i} \check{e}'_{jk} \mathcal{C}^\epsilon(s, x)) + D \nabla \mathcal{C}^\epsilon(s, x) \cdot \nabla \check{e}'_{jk}(x) \right\} dx ds \quad (2.48)$$

$$+ \epsilon Q A \int_t^{t+\delta t} \int_{\Gamma^\epsilon} \mathcal{W}^\epsilon \operatorname{div}_x^s \Upsilon^\epsilon(x) d\sigma ds \quad (2.49)$$

$$\leq C_{jk} \sqrt{\delta t}. \quad (2.50)$$

The right hand side bound (2.50) follows by virtue of the a priori estimates of Lemma 2.1. Thus, we have

$$\begin{aligned} & \left| \int_t^{t+\delta t} \frac{d}{ds} \int_{\hat{\Omega}^\epsilon(s)} \hat{T}^\epsilon(s, x) e_{jk}(x) dx ds \right| + Q \left| \int_t^{t+\delta t} \frac{d}{ds} \int_{\hat{\Omega}_g^\epsilon(s)} \hat{\mathcal{C}}^\epsilon(s, x) e'_{jk}(x) dx ds \right| \\ & \leq C_{jk} \sqrt{\delta t}. \end{aligned}$$

□

To characterize the sequence of functions $(T^\epsilon, \mathcal{C}^\epsilon)$ defined on the periodic domain Ω^ϵ in \mathbb{R}^d , we introduce extension operators, as discussed in [?, 18]. Let $E^\epsilon : H^1(\Omega^\epsilon) \rightarrow H^1(\mathbb{R}^d)$ be such that there exists a constant C , independent of ϵ , such that for all functions $\phi^\epsilon \in H^1(\Omega^\epsilon)$, with $E^\epsilon \phi^\epsilon|_{\Omega^\epsilon} = \phi^\epsilon$

$$\|E^\epsilon \phi^\epsilon\|_{L^2(\mathbb{R}^d)} \leq C \|\phi^\epsilon\|_{L^2(\Omega^\epsilon)}, \quad \|\nabla E^\epsilon \phi^\epsilon\|_{L^2(\mathbb{R}^d)} \leq C \|\nabla \phi^\epsilon\|_{L^2(\Omega^\epsilon)}. \quad (2.51)$$

To prove compactness in the moving coordinates framework, we will make use of the following sequences of functions, which are decomposed in terms of the orthonormal basis $\{e_{jk}\} \in \mathbb{R}^d$

$$\hat{T}^\epsilon(t, x) = \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}^d} \mu_{T,jk}^\epsilon(t) e_{jk}(x) \text{ with } \mu_{T,jk}^\epsilon(t) = \int_{\Omega^\epsilon(t)} \hat{T}^\epsilon(t, x) e_{jk} dx \quad (2.52)$$

$$\widehat{E^\epsilon T^\epsilon}(t, x) = \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}^d} \nu_{T,jk}^\epsilon(t) e_{jk}(x) \text{ with } \nu_{T,jk}^\epsilon(t) = \int_{\mathbb{R}^d} \widehat{E^\epsilon T^\epsilon}(t, x) e_{jk} dx. \quad (2.53)$$

Similar decompositions also hold for $\hat{\mathcal{C}}^\epsilon(t, x)$ and $\widehat{E^\epsilon \mathcal{C}^\epsilon}(t, x)$ with the corresponding time dependent Fourier coefficients $\mu_{\mathcal{C},jk}^\epsilon(t)$ and $\nu_{\mathcal{C},jk}^\epsilon(t)$ defined in $\hat{\Omega}_g^\epsilon$ and \mathbb{R}^d respectively.

Lemma 2.4. *There exists subsequences, still denoted by ϵ , such that*

$$\begin{aligned}\mu_{T,jk}^\epsilon &\rightarrow \mu_{T,jk} \text{ in } L^2(S), \quad \text{for all } j \in \mathbb{N}, k \in \mathbb{Z}^d, \\ \mu_{\mathcal{C},jk}^\epsilon &\rightarrow \mu_{\mathcal{C},jk} \text{ in } L^2(S), \quad \text{for all } j \in \mathbb{N}, k \in \mathbb{Z}^d,\end{aligned}$$

for some $\mu_{T,jk} \in L^2(S)$, respectively $\mu_{\mathcal{C},jk} \in L^2(S)$, which are the Fourier coefficients to some functions

$$T^0(t, x) = \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}^d} \mu_{T,jk}(t) e_{jk}(x), \quad \mathcal{C}^0(t, x) = \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}^d} \mu_{\mathcal{C},jk}(t) e_{jk}(x)$$

belonging to $L^2(S \times \mathbb{R}^d)$.

Proof. The idea of the proof due to [18] is as follows. A direct integration of (2.36) and (2.37) of Lemma 2.3 in time $t \in (0, T_f - \delta t)$ gives the Riesz-Fréchet-Kolmogorov (RFK) criterion for strong compactness in $L^1(S)$. Thus, for any $j \in \mathbb{N}, k \in \mathbb{Z}^d$, there exist a subsequence $\epsilon_{jk} \rightarrow 0$ and limits $\mu_{T,jk} \in L^1(S)$ and $\mu_{\mathcal{C},jk} \in L^1(S)$ such that

$$\mu_{T,jk}^{\epsilon_{jk}} \rightarrow \mu_{T,jk} \in L^1(S), \quad \mu_{\mathcal{C},jk}^{\epsilon_{jk}} \rightarrow \mu_{\mathcal{C},jk} \in L^1(S).$$

By virtue of Lemma 2.1, $\mu_{T,jk}^\epsilon$ (resp. $\mu_{\mathcal{C},jk}^\epsilon$) are bounded in $L^\infty(S)$, and the RFK property holds in $L^p(S)$, $1 \leq p < \infty$. Thus, it is straightforward to show that $(T^0, \mathcal{C}^0) \in L^2(S \times \mathbb{R}^d) \times L^2(S \times \mathbb{R}^d)$. \square

Properties of the Fourier coefficients In the next step, we estimate the difference between the Fourier coefficients μ_{jk}^ϵ and ν_{jk}^ϵ .

Lemma 2.5. *Let $\theta = |Y_g|/|Y|$. There exists a constant C_{jk} independent of ϵ such that*

$$|\mu_{T,jk}^\epsilon(t) - \nu_{T,jk}^\epsilon(t)| \leq C_{jk}\epsilon, \quad |\mu_{\mathcal{C},jk}^\epsilon(t) - \theta \nu_{\mathcal{C},jk}^\epsilon(t)| \leq C_{jk}\epsilon. \quad (2.54)$$

Proof. The definitions (2.52)-(2.53) of the Fourier coefficients, we obtain

$$\mu_{T,jk}^\epsilon(t) - \nu_{T,jk}^\epsilon(t) = \int_{\Omega^\epsilon(t)} \hat{T}^\epsilon(t, x) e_{jk}(x) dx - \theta \int_{\mathbb{R}^d} \widehat{E^\epsilon T^\epsilon}(t, x) e_{jk}(x) dx \quad (2.55)$$

$$- (1 - \theta) \int_{\mathbb{R}^d} \widehat{E^\epsilon T^\epsilon}(t, x) e_{jk}(x) dx \quad (2.56)$$

$$= \int_{\mathbb{R}^d} E^\epsilon T_s^\epsilon(t, x) \check{e}_{jk}(x) (\chi_s(x/\epsilon) - (1 - \theta)) dx \quad (2.57)$$

$$+ \int_{\mathbb{R}^d} E^\epsilon T_g^\epsilon(t, x) \check{e}_{jk}(x) (\chi_g(x/\epsilon) - \theta) dx, \quad (2.58)$$

where $\chi_i(x/\epsilon)$ are the characteristic functions of Ω_i^ϵ or their equivalents $\chi_i(y)$ defined on Y_i with $i = \{g, s\}$. To simplify (2.57)-(2.58) further, we introduce the following auxiliary problem:

$$\begin{cases} -\operatorname{div}_y(\nabla_y \Psi(y)) = \chi_s(y) - (1 - \theta), & \text{in } Y_s \\ -\operatorname{div}_y(\nabla_y \Psi(y)) = \chi_g(y) - \theta, & \text{in } Y_g \\ \Psi(y) \text{ is } Y\text{-periodic.} \end{cases} \quad (2.59)$$

Using (2.59) in (2.55) and integrating by parts the resulting expression results to

$$\left| \mu_{T,jk}^\epsilon(t) - \nu_{T,jk}^\epsilon(t) \right| \leq \epsilon \int_{\mathbb{R}^d} \left| \nabla_x \Psi^\epsilon(x) \cdot \nabla \left(E^\epsilon T_s^\epsilon(t, x) \check{e}_{jk}(x) \right) \right| dx \quad (2.60)$$

$$+ \epsilon \int_{\mathbb{R}^d} \left| \nabla_x \Psi^\epsilon(x) \cdot \nabla \left(E^\epsilon T_g^\epsilon(t, x) \check{e}_{jk}(x) \right) \right| dx \leq C_{jk} \epsilon. \quad (2.61)$$

The desired inequality in (2.61) follows by using properties (2.51) of E^ϵ and Lemma 2.1. Following a similar approach as described above leads to the second estimate of the Fourier coefficients given in (2.54). \square

Now, we state a technical result that describes how the modal series of the bounded sequences in $L^2(S; H^1(\mathbb{R}^d))$ introduced above can be truncated.

Lemma 2.6. *Let $u^\epsilon(t, x)$ be a bounded sequence in $L^2(S; H^1(\mathbb{R}^d))$. Then for any $\delta > 0$, there exists a $N(\delta)$ such that for all ϵ*

$$\left\| u^\epsilon \chi_{Q_{R(\delta)}} - \sum_{|k| \leq R(\delta)} \sum_{|j| \leq N(\delta)} \mathcal{U}_{jk}^\epsilon(t) e_{jk} \right\|_{L^2(S \times \mathbb{R}^d)} \leq \delta, \quad (2.62)$$

where $Q_{R(\delta)}$ is defined in (2.26) and

$$\mathcal{U}_{jk}^\epsilon(t) = \int_{\mathbb{R}^d} u^\epsilon(t, x) e_{jk}(x) dx \quad (2.63)$$

are the time dependent Fourier coefficients of u^ϵ defined as in (2.53).

Proof. The proof is a consequence of Lemma 2.2, the embedding $H^1(Q_{R(\delta)}) \hookrightarrow L^2(Q_{R(\delta)})$ and the a priori bounds of Lemma 2.1. For more details, we refer the interested reader to [19, Lemma 4. p. 398] and [18, Lemma 2.9]. \square

Theorem 2.2. *There exists a subsequence ϵ and limit $(T^0, \mathcal{C}^0) \in L^2(S \times \mathbb{R}^d) \times L^2(S \times \mathbb{R}^d)$ such that*

$$\lim_{\epsilon \rightarrow 0} \int \int_{S \times \hat{\Omega}^\epsilon(t)} |\hat{T}^\epsilon(t, x) - T^0(t, x)|^2 dx dt = 0, \quad (2.64)$$

$$\lim_{\epsilon \rightarrow 0} \int \int_{S \times \hat{\Omega}_g^\epsilon(t)} |\hat{\mathcal{C}}^\epsilon(t, x) - \mathcal{C}^0(t, x)|^2 dx dt = 0. \quad (2.65)$$

Proof. According to Lemma 2.2, for any $\delta > 0$ and for sufficiently large $R(\delta) > 0$, we have

$$\left\| \hat{T}^\epsilon - \hat{T}^\epsilon \chi_{Q_{R(\delta)}} \right\|_{L^2(S \times \hat{\Omega}^\epsilon(t))} \leq \frac{\delta}{5}, \quad \left\| \hat{\mathcal{C}}^\epsilon - \hat{\mathcal{C}}^\epsilon \chi_{Q_{R(\delta)}} \right\|_{L^2(S \times \hat{\Omega}_g^\epsilon(t))} \leq \frac{\delta}{5}. \quad (2.66)$$

For the extended sequences $\widehat{E^\epsilon T^\epsilon}$ (resp. $\widehat{E^\epsilon \mathcal{C}^\epsilon}$) in \mathbb{R}^d , Lemma 2.6 implies that for any $\delta > 0$, there exists $N(\delta)$ such that for $\epsilon > 0$ small enough,

$$\left\| \widehat{E^\epsilon T^\epsilon} \chi_{Q_{R(\delta)}} - \sum_{|k| \leq R(\delta)} \sum_{|j| \leq N(\delta)} \nu_{T,jk}^\epsilon(t) e_{jk}(x) \right\|_{L^2(S \times \mathbb{R}^d)} \leq \frac{\delta}{5}. \quad (2.67)$$

Without loss of generality, we will assume henceforth that similar results apply for the sequence $\hat{\mathcal{C}}^\epsilon$. Since by definition,

$$\widehat{E^\epsilon T^\epsilon} \Big|_{Q_{R(\delta)}} = \hat{T}^\epsilon \chi_{Q_{R(\delta)}}.$$

(2.67) reduces to

$$\left\| \hat{T}^\epsilon \chi_{Q_{R(\delta)}} - \sum_{|k| \leq R(\delta)} \sum_{|j| \leq N(\delta)} \nu_{T,jk}^\epsilon(t) e_{jk}(x) \right\|_{L^2(S \times \mathbb{R}^d)} \leq \frac{\delta}{5}. \quad (2.68)$$

By Lemma 2.3, for any δ and ϵ small enough, we have

$$\left\| \sum_{|k| \leq R(\delta)} \sum_{|j| \leq N(\delta)} \nu_{T,jk}^\epsilon(t) e_{jk}(x) - \sum_{|k| \leq R(\delta)} \sum_{|j| \leq N(\delta)} \mu_{T,jk}^\epsilon(t) e_{jk}(x) \right\|_{L^2(S \times \mathbb{R}^d)} \leq \frac{\delta}{5}. \quad (2.69)$$

Since by virtue of Lemma 2.4, $\mu_{T,jk}^\epsilon(t)$ is relatively compact in $L^2(S)$. Then, for sufficiently small ϵ

$$\left\| \sum_{|k| \leq R(\delta)} \sum_{|j| \leq N(\delta)} \mu_{T,jk}^\epsilon(t) e_{jk}(x) - \sum_{|k| \leq R(\delta)} \sum_{|j| \leq N(\delta)} \mu_{T,jk}(t) e_{jk}(x) \right\|_{L^2(S \times \mathbb{R}^d)} \leq \frac{\delta}{5} \quad (2.70)$$

Lemma 2.4 implies the existence of a function $T^0(t, x) \in L^2(S \times \mathbb{R}^d)$, such that for sufficiently large $N(\delta)$, we have

$$\left\| \sum_{|k| \leq R(\delta)} \sum_{|j| \leq N(\delta)} \mu_{T,jk}(t) e_{jk}(x) - T^0(t, x) \right\|_{L^2(S \times Q_{R(\delta)})} \leq \frac{\delta}{5}. \quad (2.71)$$

Finally, summing up (2.66)-(2.71), we establish (2.64), respectively (2.65). \square

3 Rigorous passage to the homogenization limit $\epsilon \rightarrow 0$

By using the concept of weak convergence methods in partial differential equations, we aim now to rediscover the structure of the macroscopic combustion equations obtained by the formal averaging by means of two-scale asymptotic expansions done in [33]. Here, we employ the mathematical tool called the two scale convergence with drift, which is a suitable modification of the two-scale convergence concept (cf. e.g. [16]) to account for suitable drifts. In this context, we follow the line of arguments from [19, 18]. Also, we refer the reader to the PhD thesis [21] as well as the references cited therein for more details and application examples of this averaging technique. We recall the definition of the two-scale convergence with drift as stated in [?,]Proposition 3.1]Harsha.

Definition 3.1. *Let b^* be a constant vector in \mathbb{R}^d and $u^\epsilon(t, x) \in L^2(S \times \mathbb{R}^d)$ be any bounded sequence of functions, i.e. there exists a constant $C > 0$, independent of ϵ , such that*

$$\|u^\epsilon\|_{L^2(S \times \mathbb{R}^d)} \leq C.$$

Then, there exists a function $u^0(t, x, y) \in L^2(S \times \mathbb{R}^d \times \mathbb{T}^d)$ and up to the extraction of a subsequence (still denoted by ϵ), the sequence u^ϵ two-scale converge with drift b^* (or equivalently in the moving coordinates $(t, x) \rightarrow (t, x - b^*t/\epsilon)$) to u^0 , in the sense that,

$$\lim_{\epsilon \rightarrow 0} \int \int_{S \times \mathbb{R}^d} u^\epsilon(t, x) \varphi\left(t, x - \frac{b^*}{\epsilon}t, \frac{x}{\epsilon}\right) dx dt = \int \int \int_{S \times \mathbb{R}^d \times \mathbb{T}^d} u^0(t, x, y) \varphi(t, x, y) dy dx dt, \quad (3.1)$$

for all test functions $\varphi(t, x, y) \in C_0^\infty(S \times \mathbb{R}^d \times \mathbb{T}^d)$.

We denote the convergence (3.1) by $u^\epsilon \xrightarrow{2\text{-drift}} u^0$. We refer to the constant vector b^* as the effective drift. At this moment its choice is arbitrary. In (ii), we will see that in our context the effective drift b^* will be played by the vectors b_T and b_C given by (3.6) and (3.7), respectively, depending whether we point out the temperature evolution or the evolution of the mass concentration.

Remark 3.1. It should be pointed out that Definition 3.1 is also applicable to a sequence of functions $u^\epsilon(t, x) \in L^2(S \times \Omega^\epsilon)$, defined in a perforated domain Ω^ϵ and satisfying the uniform bound

$$\|u^\epsilon\|_{L^2(S \times \Omega^\epsilon)} \leq C,$$

such that

$$\lim_{\epsilon \rightarrow 0} \int \int_{S \times \Omega^\epsilon} u^\epsilon(t, x) \psi\left(t, x - \frac{b^*}{\epsilon}t, \frac{x}{\epsilon}\right) dx dt = \int \int \int_{S \times \mathbb{R}^d \times Y} u^0(t, x, y) \psi(t, x, y) dy dx dt.$$

Definition 3.1 can be extended to sequences defined on periodic surfaces Γ^ϵ . In what follows, we give a statement of the two convergence with drift on periodic surfaces due to [20].

Definition 3.2. Let $b^* \in \mathbb{R}^d$ and w^ϵ be a sequence in $L^2(S \times \Gamma^\epsilon)$ such that

$$\epsilon \int \int_{S \times \Gamma^\epsilon} |w^\epsilon(t, x)|^2 d\sigma(x) dt \leq C.$$

Then, there exists a subsequence, still denoted by ϵ , and a function $w^0(t, x, y) \in L^2(S \times \mathbb{R}^d \times \Gamma)$ such that w^ϵ two-scale converge with drift b^* to $w^0(t, x, y)$ in the sense that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \epsilon \int \int_{S \times \Gamma^\epsilon} w^\epsilon(t, x) \psi\left(t, x - \frac{b^*}{\epsilon}t, \frac{x}{\epsilon}\right) d\sigma(x) dt \\ = \int \int \int_{S \times \mathbb{R}^d \times \Gamma} w^\epsilon(t, x) \psi\left(t, x - \frac{b^*}{\epsilon}t, \frac{x}{\epsilon}\right) dy d\sigma(y) dt \end{aligned}$$

for all $\psi(t, x, y) \in C_0^\infty(S \times \mathbb{R}^d; C_\#^\infty(Y))$. We denote the surface two-scale convergence with drift by $w^\epsilon \xrightarrow{2s\text{-drift}} w^0$.

Theorem 3.3. *Let b^* be a constant vector in \mathbb{R}^d and u^ϵ a sequence of functions uniformly bounded in $L^2(S; H^1(\mathbb{R}^d))$. Then there exist a subsequence, still denoted by ϵ , and functions $u^0(t, x) \in L^2(S; H^1(\mathbb{R}^d))$ and $u^1(t, x, y) \in L^2(S \times \mathbb{R}^d; H^1_\#(Y))$ such that*

$$u^\epsilon \xrightarrow{2\text{-drift}} u^0 \quad (3.2)$$

and

$$\nabla u^\epsilon \xrightarrow{2\text{-drift}} \nabla_x u^0 + \nabla_y u^1. \quad (3.3)$$

Proof. We refer the reader to [19] for the details of the proof and related context. \square

3.1 Main result

The main result in this paper is the following strong convergence result:

Theorem 3.4. *Assume (H1)–(H4). The sequence $(T^\epsilon, \mathcal{C}^\epsilon)$ of solutions to problem \mathcal{P}^ϵ satisfies*

$$T^\epsilon(t, x) = T^0(t, x - \frac{b_T}{\epsilon}t) + \rho_T^\epsilon(t, x), \quad (3.4)$$

$$\mathcal{C}^\epsilon(t, x) = \mathcal{C}^0(t, x - \frac{b_C}{\epsilon}t) + \rho_C^\epsilon(t, x), \quad (3.5)$$

with

$$\lim_{\epsilon \rightarrow 0} \int_S \int_{\Omega^\epsilon} |\rho_T^\epsilon(t, x)|^2 dt dx = \lim_{\epsilon \rightarrow 0} \int_S \int_{\Omega^\epsilon} |\rho_C^\epsilon(t, x)|^2 dt dx = 0,$$

where b_T and b_C are the effective drifts given by

$$b_T := \frac{c_g}{c^{\text{eff}}} \int_{Y_g} b(y) dy \quad (3.6)$$

and

$$b_C := \frac{1}{|Y_g|} \int_{Y_g} b(y) dy, \quad (3.7)$$

while the pair (T^0, \mathcal{C}^0) is the unique weak solution to the homogenized system

$$\begin{cases} c^{\text{eff}} \frac{\partial T^0}{\partial t} = \nabla_x \cdot (\lambda^{\text{eff}} \nabla_x T^0) & \text{in } S \times \mathbb{R}^d \\ |Y_g| \frac{\partial \mathcal{C}^0}{\partial t} = \nabla_x \cdot (D^{\text{eff}} \nabla_x \mathcal{C}^0) & \text{in } S \times \mathbb{R}^d \\ T_0(0, x) = T_0(x), \quad |Y_g| \mathcal{C}_0(0, x) = |Y_g| \mathcal{C}_0(x) & \text{in } \mathbb{R}^d. \end{cases} \quad (3.8)$$

Furthermore, the effective heat capacity is

$$c^{\text{eff}} := \int_{Y_g} c_g(y) dy + \int_{Y_s} c_s(y) dy, \quad (3.9)$$

while the entries of the dispersion tensors λ^{eff} and \mathcal{D}^{eff} are respectively given by

$$\begin{aligned} \lambda_{ij}^{\text{eff}}(T^0, \mathcal{C}^0) &:= \int_{Y_g} \lambda_g(y) (e_i + \nabla_y \chi_{g,i}) \cdot (e_j + \nabla_y \chi_{g,j}) dy \\ &+ \int_{Y_s} \lambda_s(y) (e_i + \nabla_y \chi_{s,i}) \cdot (e_j + \nabla_y \chi_{s,j}) dy \\ &+ QA \int_{\Gamma} [f'(T^0) \mathcal{C}^0 \chi_i \chi_j + f(T^0) \omega_i \chi_j] d\sigma(y), \\ i, j &= 1, \dots, d, \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} \mathcal{D}_{ij}^{\text{eff}}(T^0, \mathcal{C}^0) &:= \int_{Y_g} D(y) (e_i + \nabla_y \omega_i) \cdot (e_j + \nabla_y \omega_j) dy \\ &- A \int_{\Gamma} [f(T^0) \omega_i \omega_j + \mathcal{C}^0 f'(T^0) \omega_i \chi_j] d\sigma(y), \\ i, j &= 1, \dots, d. \end{aligned} \quad (3.11)$$

Here $(\chi, \omega) = (\chi_j, \omega_j)_{j=1, \dots, d}$ is the solution of the cell problem

$$\begin{cases} c_g(y) b(y) \cdot \nabla_y \chi_{g,j} - \nabla_y \cdot (\lambda_g(y) (\nabla_y \chi_{g,j} + e_j)) \\ = c_g(y) (b_T - b(y)) \cdot e_j, & \text{in } Y_g, \\ -\nabla_y \cdot (\lambda_s(y) (\nabla_y \chi_{s,j} + e_j)) = c_s b_T \cdot e_j, & \text{in } Y_s, \\ \chi_{g,j} - \chi_{s,i} = 0, & \text{on } \Gamma, \\ [\lambda_s(y) (\nabla_y \chi_{s,j} + e_j) - \lambda_g(y) (\nabla_y \chi_{g,j} + e_j)] \cdot n \\ = QA [f'(T^0) \mathcal{C}^0 \chi_j + f(T^0) \omega_j], & \text{on } \Gamma, \\ b(y) \cdot \nabla_y \omega_j - \nabla_y \cdot (D(y) (\nabla_y \omega_j + e_j)) = (b_C - b(y)) \cdot e_j, & \text{in } Y_g, \\ D(y) (\nabla_y \omega_j + e_j) \cdot n = -A [f'(T^0) \mathcal{C}^0 \chi_j + f(T^0) \omega_j], & \text{on } \Gamma, \\ y \rightarrow (\chi(y), \omega(y)) \text{ is } Y\text{-periodic.} \end{cases} \quad (3.12)$$

with the first-order terms written as

$$T^1(t, x, y) = \chi(y, T^0, \mathcal{C}^0) \cdot \nabla_x T^0(t, x), \quad (3.13)$$

$$\mathcal{C}^1(t, x, y) = \omega(y, T^0, \mathcal{C}^0) \cdot \nabla_x \mathcal{C}^0(t, x). \quad (3.14)$$

We prove this result in the reminder of the paper, but first we need to establish the two-scale compactness for the sequences introduced in Theorem 3.4.

Theorem 3.5. *Let $(T^\epsilon, \mathcal{C}^\epsilon)$ be a solution to \mathcal{P}^ϵ in the sense of Definition 2.1. As b^* for T^ϵ and \mathcal{C}^ϵ , we assume the constant vectors b_T and b_C defined in (3.6) and (3.7) respectively. Then, there exist subsequences $(T^\epsilon, \mathcal{C}^\epsilon)$, still denoted by ϵ , and the pairs*

$$\begin{aligned} (T^0, T^1) &\in L^2(S; H^1(\mathbb{R}^d)) \times L^2(S \times \mathbb{R}^d; H_{\#}^1(Y)), \\ (\mathcal{C}^0, \mathcal{C}^1) &\in L^2(S; H^1(\mathbb{R}^d)) \times L^2(S \times \mathbb{R}^d; H_{\#}^1(Y_g)), \end{aligned} \quad (3.15)$$

such that

$$\left\{ \begin{array}{l} T^\epsilon \xrightarrow{2\text{-drift}} T^0(t, x), \\ \mathcal{C}^\epsilon \xrightarrow{2\text{-drift}} \mathcal{C}^0(t, x), \\ \nabla T^\epsilon \xrightarrow{2\text{-drift}} \nabla_x T^0(t, x) + \nabla_y T^1(t, x, y), \\ \nabla \mathcal{C}^\epsilon \xrightarrow{2\text{-drift}} \nabla_x \mathcal{C}^0(t, x) + \nabla_y \mathcal{C}^1(t, x, y), \\ \frac{1}{\epsilon} \mathcal{C}^\epsilon f(T^\epsilon) \xrightarrow{2s\text{-drift}} \mathcal{C}^0 f'(T^0) T^1(t, x, y) + f(T^0) \mathcal{C}^1(t, x, y). \end{array} \right. \quad (3.16)$$

Proof. The statement is a straightforward Corollary of Theorem 3.3 given the ϵ -independent *a priori* estimates stated in Lemma 2.1 and Definitions 3.1 and 3.2. Therefore, the sequences stated in (3.16) have two-scale limits with drift. The identification of the limits is rather standard. For references, see e.g. [16]. However, we focus on showing how to derive the limit of the less obvious nonlinear limit in (3.16), which is part of a sequence of steps towards establishing the homogenization results stated in Theorem 3.4.

4 Proof of Theorem 3.4

We split the proof into four steps as given in the sequel.

I. Identification of the fast reaction limit In this step, we wish to handle the fast reaction term, which is potentially exploding when passing to the homogenization limit. In order to overcome this problem, inspired by a similar situation treated in [18], we proceed as follows: We take $\phi \in L^2(S \times \mathbb{R}^d \times \Gamma)$ such that

$$\int_{\Gamma} \phi(t, x, y) d\sigma(y) = 0,$$

for a.e. $(t, x) \in S \times \mathbb{R}^d$. Then, there exists a periodic vector field $\Theta \in L^2(S \times \mathbb{R}^d \times Y)^d$ with $\operatorname{div}_y \Theta = 0$ in Y and $\Theta \cdot n = \phi$ on Γ . With this auxiliary information at hand, we assume that \mathcal{C}^ϵ is extended by zero to the whole of Ω^ϵ so that

$$\begin{aligned} \epsilon \int \int_{S \times \Gamma^\epsilon} \frac{1}{\epsilon} W^\epsilon(\mathcal{C}^\epsilon, T^\epsilon) \phi(t, x - \frac{b_T}{\epsilon} t, \frac{x}{\epsilon}) d\sigma &= \int \int_{S \times \Omega^\epsilon} \operatorname{div} \left(W^\epsilon(\mathcal{C}^\epsilon, T^\epsilon) \Theta(t, x - \frac{b_T}{\epsilon} t, \frac{x}{\epsilon}) \right) dx dt \\ &= A \int \int_{S \times \Omega^\epsilon} \left(\mathcal{C}^\epsilon f'(T^\epsilon) \nabla T^\epsilon + \nabla \mathcal{C}^\epsilon f(T^\epsilon) \right) \cdot \Theta(t, x - \frac{b_T}{\epsilon} t, \frac{x}{\epsilon}) dx dt \\ &\quad + \int \int_{S \times \Omega^\epsilon} W^\epsilon(\mathcal{C}^\epsilon, T^\epsilon) \operatorname{div}_x \Theta(t, x - \frac{b_T}{\epsilon} t, \frac{x}{\epsilon}) dx dt, \end{aligned}$$

which converges as $\epsilon \rightarrow 0$ to

$$= A \int \int \int_{S \times \mathbb{R}^d \times Y} (\mathcal{C}^0 f'(T^0) (\nabla_x T^0 + \nabla_y T^1) + (\nabla_x \mathcal{C}^0 + \nabla_y \mathcal{C}^1) f(T^0)) \cdot \Theta dy dx dt$$

$$\begin{aligned}
& + A \int \int \int_{S \mathbb{R}^d Y} \mathcal{C}^0 f(T^0) \operatorname{div}_x \Theta \, dy dx dt \\
& = A \int \int \int_{S \mathbb{R}^d \Gamma} (\mathcal{C}^0 f'(T^0) T^1 + f(T^0) \mathcal{C}^1) \Theta \cdot n \, d\sigma(y) dx dt \\
& = A \int \int \int_{S \mathbb{R}^d \Gamma} (\mathcal{C}^0 f'(T^0) T^1 + f(T^0) \mathcal{C}^1) \phi \, d\sigma(y) dx dt
\end{aligned}$$

In order to obtain the last equality, we have used the periodicity in y of the vector field Θ . The convergence result relies on the essential fact that the sequence $\mathcal{C}^\epsilon f(T^\epsilon)$ has a two-scale limit, i.e. we used the strong compactness result established in Theorem 2.2 and the boundedness of $f(T) \leq T$ to deduce that $f(T^\epsilon)$ two-scale converges with drift to the limit $f(T^0)$, and hence the two-scale with drift limit $\mathcal{C}^0 f(T^0)$ of the sequence $\mathcal{C}^\epsilon f(T^\epsilon)$. \square

II. Choice of the drifts Now, we select the constant drift vector b^* such that the cell problems are weakly solvable.

Lemma 4.1. *Assume (H1)–(H3). Let $T^0(t, x) > 0$, $\mathcal{C}^0(t, x) \geq 0$ be given. Let the effective drifts b_T and b_C respectively defined as in (3.6) and (3.7) be such that there exists a unique solution*

$$(\chi, \omega) = (\chi_j, \omega_j)_{j=1, \dots, d} \in [H_\#^1(Y)]^d \times [H_\#^1(Y_g)]^d$$

to the cell problem (3.12).

Proof. One verifies directly that the corresponding compatibility condition is satisfied, i.e. by taking the average of the right hand side of (3.12) and equating the resulting expression to zero. Eventually, we deduce the definitions of the effective drift velocities given in (3.6) and (3.7). Furthermore, the coupled variational formulation of (3.12) is

$$\begin{aligned}
& \int_{Y_g} c_g b(y) \cdot \nabla_y \chi_{g,j} \phi dy + \int_{Y_g} \lambda_g (e_j + \nabla_y \chi_{g,j}) \cdot \nabla_y \phi dy + \int_{Y_s} \lambda_s (e_j + \nabla_y \chi_{s,j}) \cdot \nabla_y \phi dy \quad (4.1) \\
& + Q \int_{Y_g} b(y) \cdot \nabla_y \omega_j \psi dy + Q \int_{Y_g} D(y) (e_j + \nabla_y \omega_j) \cdot \nabla_y \psi dy \\
& + QA \int_{\Gamma} [f'(T^0) \mathcal{C}^0 \chi_j + f(T^0) \omega_j] (\psi - \phi) d\sigma(y) \\
& = \int_{Y_g} c_g (b_T - b(y)) \cdot e_j \phi dy + \int_{Y_s} c_s b_T \cdot e_j \phi dy + \int_{Y_g} (b_C - b(y)) \cdot e_j \psi dy,
\end{aligned}$$

which be shown to satisfy the assumptions of the Lax-Milgram Lemma. \square

III. Derivation of the homogenized equations and effective coefficients Take as test functions

$$(\hat{\phi}(t, x), \hat{\psi}(t, x)) = \left(\phi\left(t, x - \frac{b_T}{\epsilon} t\right), \psi\left(t, x - \frac{b_C}{\epsilon} t\right) \right),$$

with $\phi, \psi \in C_0^\infty(\mathbb{R}^d \times S)$ with $\phi(T_f, x) = \psi(T_f, x) = 0$. Also, observe that

$$\frac{\partial \phi}{\partial t}(t, x) = \frac{\partial \hat{\phi}}{\partial t}(t, x) - \frac{b_T}{\epsilon} \cdot \nabla_x \hat{\phi}(t, x)$$

and correspondingly,

$$\frac{\partial \psi}{\partial t}(t, x) = \frac{\partial \hat{\psi}}{\partial t}(t, x) - \frac{b_C}{\epsilon} \cdot \nabla_x \hat{\psi}(t, x).$$

Hence, using $(\hat{\phi}, Q\hat{\psi})$ as a test function in the coupled variational formulation of (2.13) results to

$$\begin{aligned} & \iint_{S\Omega^\epsilon} c \frac{\partial T^\epsilon}{\partial t} \hat{\phi} \, dxdt + \frac{1}{\epsilon} \iint_{S\Omega^\epsilon} c_g b^\epsilon \cdot \nabla T^\epsilon \hat{\phi} \, dxdt + \iint_{S\Omega^\epsilon} \lambda \nabla T^\epsilon \cdot \nabla \hat{\phi} \, dxdt \\ & + Q \iint_{S\Omega_g^\epsilon} \frac{\partial \mathcal{C}^\epsilon}{\partial t} \hat{\psi} \, dxdt + \frac{Q}{\epsilon} \iint_{S\Omega_g^\epsilon} b^\epsilon \cdot \nabla \mathcal{C}^\epsilon \hat{\psi} \, dxdt + Q \iint_{S\Omega_g^\epsilon} D \nabla \mathcal{C}^\epsilon \cdot \nabla \hat{\psi} \, dxdt \\ & + \frac{Q}{\epsilon} \iint_{S\Gamma^\epsilon} W^\epsilon(\hat{\psi} - \hat{\phi}) \, d\sigma^\epsilon(x) dt = 0. \end{aligned}$$

Integrating by parts, with respect to time, in the last two identities, leads to

$$\begin{aligned} & - \int_{\Omega^\epsilon} c T^\epsilon(0, x) \hat{\phi}(0, x) \, dx + \frac{1}{\epsilon} \iint_{S\Omega_g^\epsilon} c_g (b_T - b^\epsilon) T^\epsilon \nabla_x \hat{\phi} \, dxdt - \iint_{S\Omega^\epsilon} c T^\epsilon \frac{\partial \hat{\phi}}{\partial t} \, dxdt \\ & + \iint_{S\Omega^\epsilon} \lambda \nabla T^\epsilon \cdot \nabla \hat{\phi} \, dxdt - Q \int_{\Omega_g^\epsilon} \mathcal{C}^\epsilon(0, x) \hat{\psi}(0, x) \, dx + \frac{Q}{\epsilon} \iint_{S\Omega_g^\epsilon} b_C - b^\epsilon \mathcal{C}^\epsilon \nabla_x \hat{\psi} \, dxdt \\ & - Q \iint_{S\Omega_g^\epsilon} \mathcal{C}^\epsilon \frac{\partial \hat{\psi}}{\partial t} \, dxdt + Q \iint_{S\Omega_g^\epsilon} D \nabla \mathcal{C}^\epsilon \cdot \nabla \hat{\psi} \, dxdt + \frac{Q}{\epsilon} \iint_{S\Gamma^\epsilon} W^\epsilon(\hat{\psi} - \hat{\phi}) \, d\sigma^\epsilon(x) dt = 0, \end{aligned}$$

It is worth pointing out that the terms

$$\frac{1}{\epsilon} \iint_{S\Omega^\epsilon} c_g (b_T - b^\epsilon) T^\epsilon \nabla_x \hat{\phi} \, dxdt \quad \text{and} \quad \frac{Q}{\epsilon} \iint_{S\Omega_g^\epsilon} (b_C - b^\epsilon \mathcal{C}^\epsilon) \nabla_x \hat{\psi} \, dxdt$$

are potentially blowing up. Therefore, they need a special attention. To handle them, we use two auxiliary classes of vector fields Π and Σ , which we introduced earlier in (2.32) and (2.33). Thus, we obtain

$$\begin{aligned} & - \int_{\Omega^\epsilon} c T^\epsilon(0, x) \hat{\phi}(0, x) \, dx + \epsilon \iint_{S\Omega^\epsilon} \Delta \Pi_i^\epsilon \cdot \nabla_x \hat{\phi} T^\epsilon \, dxdt - \iint_{S\Omega^\epsilon} c T^\epsilon \frac{\partial \hat{\phi}}{\partial t} \, dxdt \\ & \iint_{S\Omega^\epsilon} \lambda \nabla T^\epsilon \cdot \nabla \hat{\phi} \, dxdt - Q \int_{\Omega_g^\epsilon} \mathcal{C}^\epsilon(0, x) \hat{\psi}(0, x) \, dx + \epsilon Q \iint_{S\Omega_g^\epsilon} \Delta \Sigma_i^\epsilon \cdot \nabla_x \hat{\psi} \mathcal{C}^\epsilon \, dxdt \\ & - Q \iint_{S\Omega_g^\epsilon} \mathcal{C}^\epsilon \frac{\partial \hat{\psi}}{\partial t} \, dxdt + Q \iint_{S\Omega_g^\epsilon} D \nabla \mathcal{C}^\epsilon \cdot \nabla \hat{\psi} \, dxdt + \frac{Q}{\epsilon} \iint_{S\Gamma^\epsilon} W^\epsilon(\hat{\psi} - \hat{\phi}) \, d\sigma^\epsilon(x) dt = 0, \end{aligned} \tag{4.2}$$

which gives after partial integration

$$\begin{aligned}
& - \int_{\Omega^\epsilon} cT^\epsilon(0, x) \hat{\phi}(0, x) \, dx - \iint_{S \Omega^\epsilon} cT^\epsilon \frac{\partial \hat{\phi}}{\partial t} \, dxdt + \iint_{S \Omega^\epsilon} \lambda \nabla T^\epsilon \cdot \nabla \hat{\phi} \, dxdt \\
& - \epsilon \iint_{S \Omega^\epsilon} \sum_{i=1}^d \nabla \Pi_i^\epsilon(x) \cdot \nabla \left(T^\epsilon \partial_{x_i} \hat{\phi} \right) \, dxdt + \epsilon Q \iint_{S \Omega_g^\epsilon} \sum_{i=1}^d \nabla \Sigma_i^\epsilon(x) \cdot \nabla \left(C^\epsilon \partial_{x_i} \hat{\psi} \right) \, dxdt \\
& - Q \int_{\Omega_g^\epsilon} C^\epsilon(0, x) \hat{\psi}(0, x) \, dx - Q \iint_{S \Omega_g^\epsilon} C^\epsilon \frac{\partial \hat{\psi}}{\partial t} \, dxdt + Q \iint_{S \Omega_g^\epsilon} D \nabla C^\epsilon \cdot \nabla \hat{\psi} \, dxdt \\
& + \frac{Q}{\epsilon} \iint_{S \Gamma^\epsilon} W^\epsilon(\hat{\psi} - \hat{\phi}) \, d\sigma^\epsilon(x) dt = 0,
\end{aligned} \tag{4.3}$$

Note that the terms

$$\epsilon \iint_{S \Omega^\epsilon} \sum_{i=1}^d \nabla \Pi_i^\epsilon(x) \cdot \nabla \left(T^\epsilon \partial_{x_i} \hat{\phi} \right) dxdt \quad \text{and} \quad \epsilon \iint_{S \Omega_g^\epsilon} \sum_{i=1}^d \nabla \Sigma_i^\epsilon(x) \cdot \nabla \left(C^\epsilon \partial_{x_i} \hat{\psi} \right) dxdt$$

converge in two-scale with drift to the limits

$$\iint \int_{S \mathbb{R}^d Y} \sum_{i=1}^d \nabla \Pi_i(y) \cdot \nabla_y T^1(t, x, y) \partial_{x_i} \hat{\phi}(t, x, y) dy dx dt$$

and

$$\iint \int_{S \mathbb{R}^d Y_g} \sum_{i=1}^d \nabla \Sigma_i(y) \cdot \nabla_y C^1(t, x, y) \partial_{x_i} \hat{\psi}(t, x, y) dy dx dt.$$

Now, to obtain the structure of the cell problems as well as the weak formulation of the limit equation, take as test function in (4.3) the expressions

$$\hat{\phi}(t, x) = \phi_0 \left(t, x - \frac{b_T}{\epsilon} t \right) + \epsilon \phi_1 \left(t, x - \frac{b_T}{\epsilon} t, \frac{x}{\epsilon} \right),$$

and respectively,

$$\hat{\psi}(t, x) = \psi_0 \left(t, x - \frac{b_C}{\epsilon} t \right) + \epsilon \psi_1 \left(t, x - \frac{b_C}{\epsilon} t, \frac{x}{\epsilon} \right),$$

where $(\phi_0(t, x), \phi_1(t, x, y)) \in C_0^\infty(S \times \mathbb{R}^d) \times C_0^\infty(S \times \mathbb{R}^d; H_\#^1(Y))$

and $(\psi_0(t, x), \psi_1(t, x, y)) \in C_0^\infty(S \times \mathbb{R}^d) \times C_0^\infty(S \times \mathbb{R}^d; H_\#^1(Y_g))$ and satisfy

$$\phi_i(T_f, x) = \psi_i(T_f, x) = 0. \tag{4.4}$$

In [18], it was necessary to assume the same variable for the zeroth order terms of the oscillating test functions, in order to deduce the desired coupling of the single physics problem at the

macroscopic level. However, in the present multi-physics scenario, we assume the restriction of the zeroth order terms of the test functions on Γ^ϵ to be equal, i.e.,

$$\phi_0(t, x) \Big|_{\Gamma^\epsilon} = \psi_0(t, x) \Big|_{\Gamma^\epsilon} \text{ for a.e. } t \in S. \quad (4.5)$$

It is now important to take as first step $\phi_0 = \psi_0 = 0$ and then pass to the two-scale limit with drift in (4.3). We obtain the following

$$\begin{aligned} & \int \int \int_{S \mathbb{R}^d Y} c(y) T^0 b_T \cdot \nabla_x \phi_1 dy dx dt + \int \int \int_{S \mathbb{R}^d Y_g} c_g b(y) \cdot (\nabla_x T^0 + \nabla_y T^1) \phi_1 dy dx dt \\ & + \int \int \int_{S \mathbb{R}^d Y} \lambda(y) (\nabla_x T^0 + \nabla_y T^1) \cdot \nabla_y \phi_1 dy dx dt \\ & - QA \int \int \int_{S \mathbb{R}^d \Gamma} (\mathcal{C}^0 f'(T^0) T^1 + f(T^0) \mathcal{C}^1) \phi_1 d\sigma(y) dx dt = 0. \end{aligned} \quad (4.6)$$

Similarly, we obtain

$$\begin{aligned} & \int \int \int_{S \mathbb{R}^d Y_g} \mathcal{C}^0 b_C \cdot \nabla_x \psi_1 dy dx dt + \int \int \int_{S \mathbb{R}^d Y_g} b(y) \cdot (\nabla_x \mathcal{C}^0 + \nabla_y \mathcal{C}^1) \psi_1 dy dx dt \\ & + \int \int \int_{S \mathbb{R}^d Y_g} D(y) (\nabla_x \mathcal{C}^0 + \nabla_y \mathcal{C}^1) \cdot \nabla_y \psi_1 dy dx dt \\ & + A \int \int \int_{S \mathbb{R}^d \Gamma} (\mathcal{C}^0 f'(T^0) T^1 + f(T^0) \mathcal{C}^1) \psi_1 d\sigma(y) dx dt = 0. \end{aligned} \quad (4.7)$$

(4.6) and (4.7) are simply the variational formulation of

$$\left\{ \begin{array}{ll} -c_g b_T \cdot \nabla_x T^0 + c_g b(y) \cdot (\nabla_x T^0 + \nabla_y T^1) & \text{in } Y_g \\ -\operatorname{div}_y (\lambda_g(y) (\nabla_x T^0 + \nabla_y T^1)) = 0, & \text{in } Y_g \\ -c_s b_T \cdot \nabla_x T^0 - \operatorname{div}_y (\lambda_s(y) (\nabla_x T^0 + \nabla_y T^1)) = 0, & \text{in } Y_s \\ [T^1]_\Gamma = 0, & \text{on } \Gamma \\ [\lambda_s (\nabla_x T^0 + \nabla_y T^1) - \lambda_g (\nabla_x T^0 + \nabla_y T^1)] \cdot n & \text{on } \Gamma \\ = QA(\mathcal{C}^0 f'(T^0) T^1 + f(T^0) \mathcal{C}^1), & \text{on } \Gamma \\ -b_C \cdot \nabla_x \mathcal{C}^0 + b(y) \cdot (\nabla_x \mathcal{C}^0 + \nabla_y \mathcal{C}^1) & \text{in } Y_g \\ -\operatorname{div}_y (D(y) (\nabla_x \mathcal{C}^0 + \nabla_y \mathcal{C}^1)) = 0, & \text{in } Y_g \\ D(y) (\nabla_x \mathcal{C}^0 + \nabla_y \mathcal{C}^1) \cdot n = -A(\mathcal{C}^0 f'(T^0) T^1 + f(T^0) \mathcal{C}^1), & \text{on } \Gamma \\ (T^1(y), \mathcal{C}^1(y)) \text{ is } Y\text{-periodic.} & \end{array} \right. \quad (4.8)$$

(4.8) implies, by linearity argument, that

$$T^1(t, x, y) = \sum_{i=1}^d \chi_i(y) \frac{\partial T^0}{\partial x_i}(t, x),$$

and

$$\mathcal{C}^1(t, x, y) = \sum_{i=1}^d \omega_i(y) \frac{\partial \mathcal{C}^0}{\partial x_i}(t, x),$$

where (χ_i, ω_i) , $i = 1, \dots, d$ solves the cell problem (3.12). As second and last step, we take $\phi_1 = \psi_1 = 0$ and pass again to the two-scale limit with drift in (4.3). This results to the following limit equations:

$$\iint_{S \mathbb{R}^d} c^{\text{eff}} \frac{\partial T^0}{\partial t} \phi_0 + \sum_{i,j=1}^d \iiint_{S \mathbb{R}^d Y} \lambda_{ij}(y) \frac{\partial T^0}{\partial x_j} \frac{\partial \phi_0}{\partial x_i} dy dx dt \quad (4.9)$$

$$+ \iiint_{S \mathbb{R}^d Y} \sum_{i,j=1}^d \sum_{l=1}^d \lambda_{il}(y) \frac{\partial \chi_j(y)}{\partial y_l} \frac{\partial T^0}{\partial x_j} \frac{\partial \phi_0}{\partial x_i} dy dx dt$$

$$+ \iiint_{S \mathbb{R}^d Y} \sum_{i,j=1}^d \sum_{l=1}^d \frac{\partial \Pi_i(y)}{\partial y_l} \frac{\partial \chi_j(y)}{\partial y_l} \frac{\partial T^0}{\partial x_j} \frac{\partial \phi_0}{\partial x_i} dy dx dt$$

$$+ Q \left\{ \iint_{S \mathbb{R}^d} |Y_g| \frac{\partial \mathcal{C}^0}{\partial t} \psi_0 + \sum_{i,j=1}^d \iiint_{S \mathbb{R}^d Y_g} D_{ij}(y) \frac{\partial \mathcal{C}^0}{\partial x_j} \frac{\partial \psi_0}{\partial x_i} dy dx dt \quad (4.10)$$

$$+ \iiint_{S \mathbb{R}^d Y_g} \sum_{i,j=1}^d \sum_{l=1}^d D_{il}(y) \frac{\partial \omega_j(y)}{\partial y_l} \frac{\partial \mathcal{C}^0}{\partial x_j} \frac{\partial \psi_0}{\partial x_i} dy dx dt$$

$$+ \iiint_{S \mathbb{R}^d Y_g} \sum_{i,j=1}^d \sum_{l=1}^d \frac{\partial \Sigma_i(y)}{\partial y_l} \frac{\partial \omega_j(y)}{\partial y_l} \frac{\partial \mathcal{C}^0}{\partial x_j} \frac{\partial \psi_0}{\partial x_i} dy dx dt \Big\} = 0,$$

which is nothing but the variational formulation of the homogenized problem (3.8) and the effective dispersion tensors are given, according to (4.9) and (4.10), by

$$\lambda_{ij}^{\text{eff}} = \int_Y \lambda(y) e_i \cdot e_j dy + \int_Y \lambda(y) \nabla_y \chi_j \cdot e_i dy + \int_Y \nabla_y \Pi_i \cdot \nabla_y \chi_j dy \quad (4.11)$$

and

$$\mathcal{D}_{ij}^{\text{eff}} = \int_{Y_g} D(y) e_i \cdot e_j dy + \int_{Y_g} D(y) \nabla_y \omega_j \cdot e_i dy + \int_{Y_g} \nabla_y \Sigma_i \cdot \nabla_y \omega_j dy, \quad (4.12)$$

while c^{eff} is defined by (3.9). In (4.9) and (4.10), we identify the solutions to the auxiliary problems given in (2.32), which simplify the singular terms in the variational formulations (2.15) and (2.16). We shall employ (2.32) presently in transforming the dispersion matrix (4.11), hence proving that it is equivalent to the formula (3.10). To achieve this, we test (2.32) for Π_i by the cell solution χ_j . Adding the resulting expressions lead to

$$\int_Y \nabla_y \Pi_i \cdot \nabla_y \chi_j(y) dy = \int_Y c(y) (b_{T,i} - b_i(y)) \chi_j(y) dy. \quad (4.13)$$

Substituting (4.13) in (4.11) gives the non-symmetrized form of the dispersion tensor obtained by means of formal asymptotics technique in [33]. In a next step, we replace the test functions in the variational formulation of (3.12) for (χ_i, ω_i) by (χ_j, ω_j) . This results to the following

$$\begin{aligned} \int_Y c(y)(b_{T,i} - b_i(y)) \chi_j(y) dy &= \int_Y \lambda(y) \nabla_y \chi_i \cdot \nabla_y \chi_j dy \\ &+ \int_Y \lambda(y) \nabla_y \chi_j \cdot e_i dy + QA \int_{\Gamma} \left(f'(T^0) \mathcal{C}^0 \chi_i + f(T^0) \omega_i \right) \chi_j(y) d\sigma(y). \end{aligned} \quad (4.14)$$

Eventually, after substituting (4.14) in (4.11), we see that both formulas (3.10) and (4.11) for the dispersion tensor λ^{eff} are equivalent. A similar argument leads to the dispersion tensor \mathcal{D}^{eff} . Thus, we have obtained the variational formulation of the homogenized problem (3.8), which, according to Lemma 4.2, admits a unique solution. It should be noted that as a consequence of the uniqueness of the solutions, the entire sequence converges.

IV. Uniqueness of solutions to the homogenized equations

Lemma 4.2. *Under the assumptions of Lemma 4.1, there exists a unique solution to the couples*

$$(T^0, \mathcal{C}^0) \in C(\bar{S}; L^2(\mathbb{R}^d)) \times C(\bar{S}; L^2(\mathbb{R}^d))$$

and

$$(\nabla T^0, \nabla \mathcal{C}^0) \in L^2(S \times \mathbb{R}^d) \times L^2(S \times \mathbb{R}^d)$$

Proof. By construction, the symmetric part of the dispersion tensor satisfying (4.1) is given by

$$\begin{aligned} \mathfrak{L}_{ij}^{\text{sym}}(T^0, \mathcal{C}^0) &= \int_{Y_g} \lambda_g(y) (e_i + \nabla_y \chi_{g,i}) \cdot (e_j + \nabla_y \chi_{g,j}) dy \\ &+ \int_{Y_s} \lambda_s(y) (e_i + \nabla_y \chi_{s,i}) \cdot (e_j + \nabla_y \chi_{s,j}) dy \\ &+ Q \int_{Y_g} D(y) (e_i + \nabla_y \omega_i) \cdot (e_j + \nabla_y \omega_j) dy \\ &+ QA \left(f(T^0) - f'(T^0) \mathcal{C}^0 \right) \int_{\Gamma} \frac{(\omega_j \chi_i + \omega_i \chi_j)}{2} d\sigma(y) \\ &+ f'(T^0) \mathcal{C}^0 \int_{\Gamma} \chi_i \chi_j d\sigma(y) - f(T^0) \int_{\Gamma} \omega_i \omega_j d\sigma(y). \end{aligned} \quad (4.15)$$

Since $f(T^0) \leq T^0$ and $Q > 0$, we have

$$\mathcal{D}^{\text{eff}}(T^0, \mathcal{C}^0) \leq \int_{Y_g} D(y) dy, \quad \lambda^{\text{eff}}(T^0, \mathcal{C}^0) \leq C,$$

for some $C \in (0, \infty)$ and hence $\mathcal{D}^{\text{eff}}(T^0, \mathcal{C}^0)$ and $\lambda^{\text{eff}}(T^0, \mathcal{C}^0)$ are uniformly bounded. Given that the diffusion tensors (3.10) and (3.11) are symmetric and $f(T^0) \geq 0$, we have that

$$\lambda^{\text{eff}}(T^0, \mathcal{C}^0) \geq \left[\int_{Y_g} \lambda_g(y) dy + \int_{Y_s} \lambda_s(y) dy \right] > \lambda_0, \quad (4.16)$$

for some $\lambda_0 \in (0, \infty)$. (4.15) also implies that

$$\mathfrak{L}(T^0, \mathcal{C}^0) \geq \int_{Y_g} \lambda_g(y) dy + \int_{Y_s} \lambda_s(y) dy,$$

for $Q > 0$ and $f(T^0) \geq 0$. Using the fact that $\mathfrak{L} = \lambda^{\text{eff}} + Q\mathcal{D}^{\text{eff}}$ and the estimate (4.16) with

$$\lambda_0 = \max \left(\int_{Y_g} \lambda_g(y) dy, \int_{Y_s} \lambda_s(y) dy \right),$$

then $\mathcal{D}^{\text{eff}}(T^0, \mathcal{C}^0)$ is bounded from below by

$$\mathcal{D}^{\text{eff}}(T^0, \mathcal{C}^0) \geq \frac{1}{Q} \left[\int_{Y_g} \lambda_g(y) dy + \int_{Y_s} \lambda_s(y) dy - \lambda_0 \right]. \quad (4.17)$$

Thus, $\lambda^{\text{eff}}(T^0, \mathcal{C}^0)$, respectively $\mathcal{D}^{\text{eff}}(T^0, \mathcal{C}^0)$, is uniformly coercive and hence the uniqueness follows by standard arguments for parabolic equations. \square

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